UNSTEADY MHD FREE CONVECTION COUETTE FLOW BETWEEN TWO VERTICAL PERMEABLE PLATES IN THE PRESENCE OF THERMAL RADIATION

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Abstract: The unsteady of MHD free convection Coutte flow between two vertical permeable plates are studied in the presence of thermal radiation. The different parameters values are shown in graphically.

Introduction:

The study of magneto hydrodynamics (MHD) is of great importance in many engineering applications such as in the use of MHD pumps in chemical energy technology and in the use of MHD power generators.

The study of MHD Couette flow is important in industrial and engineering applications such as MHD pumps, MHD power generators, polymer technology and electrostatic precipitation extended the work of Hartmann and Lazarus to the case of unsteady Couette flow. further extended this study by considering the Couette flow in the presence of uniform suction and injection between porous walls when one of the walls is uniformly accelerated.

The unsteady MHD Couette flow between two infinite parallel porous plates with uniform suction and injection in the cases of impulsive and uniformly accelerated movement of the lower plate was studied by Seth. The lower plate is at rest in both cases and the magnetic field is fixed with respect to the moving plate. The velocity distribution and the shear stress on the moving plate were obtained using the Laplace transform technique. It was observed that in both cases, increasing the magnetic parameter results in an increase in the velocity, and increasing the suction parameter decreases the velocity.

This paper aims to extend the work of Rajput and Sahu by incorporating the effects of uniform suction and injection through the plates. The problem is solved using Galerkin's finite element method and the effects of suction parameter S, radiation parameter Rd, Grash of number Gr, magnetic parameter H and Prandtl number Pr on both the velocity and temperature distributions are investigated.

MATHEMATICAL ANALYSIS

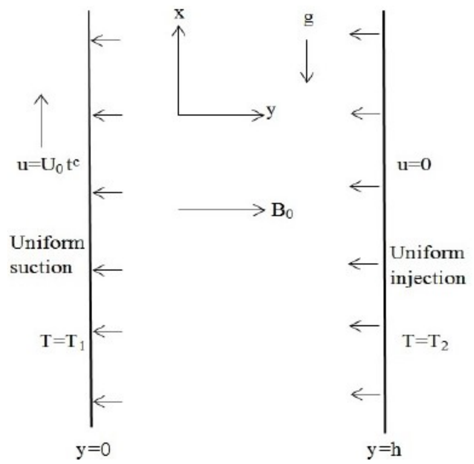


Figure 1: Schematic Diagram of the Physical System

The incompressible Newtonian fluid flows between two parallel vertical non-conducting permeable plates. These plates are located on planes y = 0 and y = h, and are infinite in the x and z directions. The plate at y = h is stationary and the other plate moves with time-dependent velocity U_0t^c in the positive x-direction (where U_0 is constant and cis a non-negative integer). The temperature of the moving and stationary plates are fixed at T_1 and T_2 respectively, with $T_1>T_2$. Uniform suction through the moving plate and uniform injection through the stationary plate are applied through the plates at t = 0 in the negative y direction. A magnetic field with magnitude B0, which is fixed relative to the moving plate, is applied in the positive y-direction.

We make the following simplifying assumptions.

- The magnetic Reynolds number is very small.
- For a typical conductor, the charge density $\rho_e$is very small; hence it is negligible.
- The Boussinesq approximation is applied.
- The fluid is a gray and optically thick absorbing-emitting but non-scattering medium.
- The fluid has a refractive index of unity.

We have considered the flow of unsteady viscous incompressible fluid. The x – axis is taken along the plate in the upward direction and y- axis is taken normal to the plate. Initially the fluid and plate are at the same temperature. A transverse magnetic field B_0, of uniform strength is applied normal to the plate. The viscous dissipation and induced magnetic field has been neglected due to its small effect. Initially, the fluid and plate are at the same temperature T_∞ and concentration C_∞ in the stationary condition. At time t > 0, the plate is moving with a velocity $u = u_0$ in its own plane and the temperature of the plate is raised to T_w and the concentration level near the plate is raised linearly with respect to time. The flow modal is as under:

$$\frac{\partial u}{\partial t} = g\beta(T - T_\infty) + g\beta(C - C_\infty) + v\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho}, \tag{1}$$

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial y^2} - \frac{\partial qr}{\partial y}, \tag{2}$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial y^2}, \tag{3}$$

With the following initial and boundary conditions :

$$\left.\begin{array}{l} \left.\begin{array}{r} u=0 \\ t\le 0: T=T_\infty, \\ C=C_\infty, \end{array}\right\} \text{for all the values of } y \\ \left.\begin{array}{l} u=u_0, \\ t>0: \begin{array}{l} \left.\begin{array}{l} T=T_\infty+(T_w=T_\infty)At, \\ C=C_\infty+(C_w=C_\infty)At, \end{array}\right\} at\ y=0 \end{array} \end{array}\right. \\ \left.\begin{array}{l} u\to 0, \\ T\to T_\infty \\ C\to C_\infty \end{array}\right\} as\ y\to 0 \end{array}\right\} \tag{4}$$

Where $A=\frac{u_0^2}{v}$

The local radiant for the case of an optically thin gray gas is expressed by

$$\frac{\partial qr}{\partial y}=-4a*\sigma\left(T_\infty^4-T^4\right), \tag{5}$$

Considering the temperature difference within the flow sufficiently small, T^4 can be expressed as the linear function of temperature. This is accomplished by expanding T^4 in a Taylor series about T_∞ and neglecting higher-order, thus

$$T^4\cong 4T_\infty^3T-3T_\infty^4 \tag{6}$$

Using equations (5) and (6), equation (2) reduces to

$$\rho C_p\frac{\partial^2 T}{\partial y^2}+16a*\sigma T_\infty^3(T_\infty-T) \tag{7}$$

Introducing the following non-dimensional quantities:

$$\left.\begin{array}{lll} \bar{u}=\frac{u}{u_0}, & \bar{y}=\frac{yu_0}{v}, & \theta=\frac{(T-T_\infty)}{(T_w-T_\infty)}, \\ G_r=\frac{g\beta*v(T_w-T_\infty)}{u_0^3}, & P_r=\frac{\mu C_p}{k}, & \bar{C}=\frac{(C-C_\infty)}{(C_w-C_\infty)}, \\ G_m=\frac{g\beta*v(C_w-C_\infty)}{u_0^3}, & S_c=\frac{v}{D}, & M=\frac{\sigma B_0^2 v}{\rho u_0^2}, \\ R=\frac{16a*v^2\sigma T_\infty^3}{ku_0^2}, & \bar{t}=\frac{tu_0^2}{v}\ and & \mu=\rho v. \end{array}\right\} \tag{8}$$

Equations (1), (2) and (7) leads to

$$\frac{\partial \bar{u}}{\partial \bar{t}} = G_r\theta + G_m\bar{C} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - M\bar{u}\,, \tag{9}$$

$$\frac{\partial \theta}{\partial \bar{t}} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial \bar{y}^2} - \frac{R}{P_r}\theta\,, \tag{10}$$

$$\frac{\partial \bar{C}}{\partial \bar{t}} = \frac{1}{S_c}\frac{\partial^2 \bar{C}}{\partial \bar{y}^2} \tag{11}$$

The initial and boundary conditions in dimensionless form are as follows :

$$\left.\begin{array}{l} \bar{t} \le 0 : \left.\begin{array}{l} \bar{u} = 0 \\ \theta = 0, \\ \bar{C} = 0, \end{array}\right\} \textit{for all the values of } \bar{y} \\ \bar{t} > 0 : \left.\begin{array}{l} \bar{u} = 1, \\ \theta = \bar{t}, \\ \bar{C} = \bar{t}, \end{array}\right\} at\ \bar{y} = 0 \\ \qquad \left.\begin{array}{l} \bar{u} \to 0, \\ \theta \to 0 \\ \bar{C} \to 0 \end{array}\right\} as\ \bar{y} \to \infty \end{array}\right\} \tag{12}$$

Dropping bars in the above equations, we have,

$$\frac{\partial u}{\partial t} = G_r\theta + G_m C + \frac{\partial^2 u}{\partial y^2} - Mu \tag{13}$$

$$\frac{\partial \theta}{\partial \bar{t}} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2} - \frac{R}{P_r}\theta\,, \tag{14}$$

$$\frac{\partial C}{\partial t} = \frac{1}{S_c}\frac{\partial^2 C}{\partial y^2} \tag{15}$$

With boundary conditions :

$$\left.\begin{array}{l} t \le 0 : \left.\begin{array}{l} u(y,t)=0 \\ \theta(y,t)=0, \\ C(y,t)=0 \end{array}\right\} \text{for all the values of } y \\ t > 0 : \left.\begin{array}{l} u(y,t)=1, \\ \theta(y,t)=t, \\ C(y,t)=t, \end{array}\right\} at\ y=0, \\ \left.\begin{array}{l} u(y,t)\to 0, \\ \theta(y,t)\to 0 \\ C(y,t)\to 0 \end{array}\right\} as\ y\to\infty \end{array}\right\} \tag{16}$$

The dimensionless governing equations (13) to (15), subject to the boundary conditions (16), are solved by the usual Laplace transform technique. Help of Hetnarski's (1975) development has also been taken.

The solutions derived are given below.

Transforming equation (15) we get,

$$L\left(\frac{\partial C(y,t)}{\partial t}\right) = L\left(\frac{1}{S_c}\frac{\partial^2 C(y,t)}{\partial y^2}\right);$$

i.e. $$sL(C(y,t)) - C(y,0) = \frac{1}{S_c} L\left(\frac{\partial^2 C(y,t)}{\partial y^2}\right) \tag{17}$$

using boundary conditions (16), we have,

$$sL(C(y,t)) - \frac{1}{S_c}\frac{\partial^2 L(C(y,t))}{\partial y^2},$$

or

$$\frac{\partial^2 L(C(y,t))}{\partial y^2} - sS_c L(C(y,t)) = 0$$

It's solution will be :

$$L(C(y,t)) = A_1 e^{y\sqrt{sS_c}} + A_2 e^{-y\sqrt{sS_c}}, \tag{18}$$

Where A_1 and A_2 are arbitrary constants.

Again using above boundary conditions (16), we get,

$$L(C(y,t)) = L(t) = \frac{1}{s^2} = A_1 + A_2 \tag{19}$$

Now, since C(y, t) must be bounded as y → ∞, we must have C(y, s) also bounded as y → ∞ and it follows that we must choose A_1 (Spieget (1986) page no.-97).

$$\therefore L(C(y,t)) = L(0) = 0 = A_1, \tag{20}$$

Using equations (19) and (20), equation (18) reduces to:

$$L(C(y,t)) = \frac{1}{s^2} e^{-y\sqrt{sS_c}} \tag{21}$$

It implies $C(y,t) = L^{-1}\left(\frac{1}{s^2} e^{-y\sqrt{sS_c}}\right)$, from Campbell G A and Foster R M (1948), we get,

$$\therefore C(y,t) = t\left[\left(1 + 2\eta^2 S_c\right) erfc\left(\eta\sqrt{S_c}\right) - \frac{2\eta\sqrt{S_c}}{\sqrt{\pi}} e^{-\eta^2 S_c}\right] \tag{22}$$

Where $\eta = \frac{y}{2\sqrt{t}}$

Also transforming equation (14);

$$L\left(\frac{\partial \theta(y,t)}{\partial t}\right) = L\left(\frac{1}{P_r}\frac{\partial^2 \theta(y,t)}{\partial y^2} - \frac{R}{P_r}\theta(y,t)\right);$$

i.e. $$sL(\theta(y,t)) - \theta(0,y) = \frac{1}{P_r}\frac{d^2 L(\theta(y,t))}{dy^2} - \frac{R}{P_r} L(\theta(y,t)), \tag{23}$$

Using boundary conditions (16), it reduces to :

$$\frac{d^2 L(\theta(y,t))}{dy^2} - (R + sP_r)L(\theta(y,t)) = 0$$

It's final solution under (16) will be,

$$L(\theta(y,t)) = A_3 e^{y\sqrt{R+sP_r}} + A_4 e^{-y\sqrt{R+sP_r}},$$

Where A_3 and A_4 are arbitrary constants.

Values of A_3 and A_4 can be computed using (16);

 UGC CARE Listed (Group -I) Journal Volume 11, Iss 10, 2022

i.e. $A_3 = 0$ and $A_4 = \frac{1}{s^2}$.

Therefore,

$$L(\theta(y,t)) = \frac{1}{s^2} e^{-y\sqrt{P_r}\sqrt{s+c}}, \tag{24}$$

Where $c = \frac{R}{P_r}$.

From this we can obtain θ(y, t),

i.e. $\theta(y,t) = L^{-1}\left(\frac{1}{s^2} e^{-y\sqrt{P_r}\sqrt{s+c}}\right)$,

$$\therefore \theta(y,t) = \frac{t}{2}\left[\theta_1 e^{2\eta\sqrt{Rt}} erfc\left(\eta\sqrt{P_r} + \sqrt{ct}\right) + \theta_2 e^{-2\eta\sqrt{Rt}} erfc\left(\eta\sqrt{P_r} - \sqrt{ct}\right)\right], \tag{25}$$

Where $\theta_1 = \left(1 + \frac{\eta P_r}{\sqrt{Rt}}\right)$ and $\theta_2 = \left(1 - \frac{\eta P_r}{\sqrt{Rt}}\right)$.

Again transforming equation (13);

$$L\left(\frac{\partial u(y,t)}{\partial t}\right) = L\left(G_r\theta(y,t) + G_m C(y,t) + \frac{\partial^2 u(y,t)}{\partial y^2} - Mu(y,t)\right),$$

We get

$$sL(u(y,t)) - u(y,t) = G_r L(\theta(y,t)) + G_m L(C(y,t)) + \frac{d^2 L(u(y,t))}{dy^2} - ML(u(y,t)) \tag{26}$$

Applying boundary conditions (16), we get,

$$\frac{d^2 L(u(y,t))}{dy^2} - (s+M)L(u(y,t)) = -G_r L(\theta(y,t)) - G_m L(C(y,t))$$

It's general solution will be :

$$L(u(y, t)) = A.E. + P.I., \tag{27}$$

With

$$A.E. = A_5 e^{y\sqrt{s+M}} + A_6 e^{-y\sqrt{s+M}},$$

$$P.I. = -G_r \frac{1}{(D^2 - s + M)} L(\theta(y,t)) - G_m \frac{1}{(D^2 - s + M)} L(C(y,t)),$$

Where A_5, A_6 are arbitrary constants.

Substituting values of L(C(y, t) and L(θ(y, t)) from equations (21) and (24) respectively, equation (27) becomes,

$$L(u(y,t)) = A_5 e^{y\sqrt{s+M}} + A_6 e^{-y\sqrt{s+M}} + \frac{G}{(1-P_r)} \frac{e^{-y\sqrt{P_r}\sqrt{s+c}}}{s^2(s-a)} + \frac{G_m}{(1-S_c)} \frac{e^{-y\sqrt{sS_c}}}{s^2(s-b)},$$

Where $a = \frac{R-M}{1-P_r}$ and $b = \frac{M}{S_c - 1}$ (28)

For finding the values of A_5 and A_6, we apply boundary conditions (16).

We obtain,

$A_5 = 0$ and $A_6 = \frac{1}{8} - \frac{G_r}{(1-P_r)} \frac{1}{s^2(s-a)} - \frac{G_m}{(1-S_c)} \frac{1}{s^2(s-b)}$

Therefore equation (24) becomes :

$$L(u(y,t)) = \frac{1}{s} e^{-y\sqrt{s+M}} + \frac{G_r}{1-P_r} \left(\frac{e^{-y\sqrt{P_r}+\sqrt{s+c}} - e^{-y\sqrt{s+M}}}{s^2(s-a)} \right) + \frac{G_m}{1-S_c} \left(\frac{e^{-y\sqrt{sS_c}} - e^{-y\sqrt{s+M}}}{s^2(s-b)} \right) \quad (29)$$

This gives,

$$u(y,t) = L^{-1}\left[\frac{1}{s} e^{-y\sqrt{s+M}} + \frac{G_r}{1-P_r} \left(\frac{e^{-y\sqrt{P_r}+\sqrt{s+c}} - e^{-y\sqrt{s+M}}}{s^2(s-a)} \right) + \frac{G_m}{1-S_c} \left(\frac{e^{-y\sqrt{sS_c}} - e^{-y\sqrt{s+M}}}{s^2(s-b)} \right) \right],$$

$$\therefore u(y,t) = B_1 e^{-y\sqrt{M}} erfc\left(\eta - \sqrt{Mt}\right) + B_2 e^{y\sqrt{M}} erfc\left(\eta + \sqrt{Mt}\right)$$

$$+ G_5\left[e^{-y\sqrt{P_r}\sqrt{a+c}} erfc\left(\eta\sqrt{P_r} - \sqrt{(a+c)t}\right) + e^{y\sqrt{P_r(a+c)}} erfc\left(\eta\sqrt{P_r} + \sqrt{(a+c)t}\right) \right]$$

$$- \left[B_3 e^{-y\sqrt{cP_r}} erfc\left(\eta\sqrt{P_r} - \sqrt{ct}\right) + B_4 e^{y\sqrt{cP_r}} erfc\left(\eta\sqrt{P_r} + \sqrt{ct}\right) \right]$$

$$+ G_5\left[e^{-y\sqrt{M_1 P_r}} erfc\left(\eta\sqrt{P_r} - \sqrt{M_1 t}\right) + e^{y\sqrt{M_1 P_r}} erfc\left(\eta\sqrt{P_r} + \sqrt{M_1 t}\right) \right]$$

$$+ G_6\left[e^{-y\sqrt{bS_c}} erfc\left(\eta\sqrt{S_c} - \sqrt{bt}\right) + e^{y\sqrt{bS_c}} erfc\left(\eta\sqrt{S_c} + \sqrt{bt}\right) \right]$$

$$- G_6\left[e^{-y\sqrt{M_2}}\, erfc\left(\eta - \sqrt{M_2 t}\right) + e^{y\sqrt{M_2}}\, erfc\left(\eta + \sqrt{M_2 t}\right)\right]$$

$$- G_4\left\{1 + bt\left(1 + 2\eta^2 S_c\right) erfc\left(\eta\sqrt{S_c}\right) + \frac{2bt\eta\sqrt{S_c}}{\sqrt{\pi}} e^{-\eta^2 S_c}\right\}, \qquad (30)$$

For making the equation (30) concise, the following symbols have been introduced:

$$B_1 = \frac{1}{2}\left[G_1 + G_2\left(t - \frac{y}{2\sqrt{M}}\right)\right], \qquad B_2 = \frac{1}{2}\left[G_1 + G_2\left(t + \frac{y}{2\sqrt{M}}\right)\right],$$

$$B_3 = \frac{G_3}{2}\left[1 + at - \frac{ya\sqrt{P_r}}{2\sqrt{c}}\right], \qquad B_4 = \frac{G_3}{2}\left[1 + at + \frac{ya\sqrt{P_r}}{2\sqrt{c}}\right],$$

$$G_1 = 1 + G_3 + G_4, \qquad G_2 = \frac{G_r}{a(1 - P_r)} + \frac{G_m}{b(1 - S_c)},$$

$$G_3 = \frac{G_r}{a^2(1 - P_r)}, \qquad G_4 = \frac{G_m}{b^2(1 - S_c)},$$

$$G_5 \frac{G_3}{2} e^{at}, \qquad G_6 \frac{G_4}{2} e^{bt},$$

$M_1 = M + a$ and $M_2 = M + b$

3 SKIN FRICTION

The non-dimensional form of skin friction is given by:

$$\tau = -\left(\frac{\partial u(y,t)}{\partial y}\right)_{y=0} = -\frac{1}{2\sqrt{t}}\left(\frac{\partial u(y,t)}{\partial \eta}\right)_{\eta=0}, \qquad (31)$$

Therefore, we get,

$$\tau = \left(t\sqrt{M} + \frac{1}{2\sqrt{M}}\right) G_2\, erf\left(\sqrt{Mt}\right) + \frac{tG_2}{\sqrt{M\pi}} e^{-Mt} - G_3\left[M_1 e^{-at}\, erf\left(\sqrt{M_1 t}\right) + \frac{a\sqrt{tP_r}}{\sqrt{\pi}} e^{-ct}\right]$$

$$- G_3\sqrt{P_r}\left[\frac{(a + 2c + 2act)}{2\sqrt{t}}\, erf\left(\sqrt{ct}\right) - e^{at}\sqrt{(a+c)}\, erf\left(\sqrt{(a+c)t}\right)\right]$$

$$- G_4 e^{bt}\left[\sqrt{M_2}\, erf\left(\sqrt{M_2 t}\right) - \frac{bS_c}{2}\, erf\left(\sqrt{bt}\right) - \frac{e^{-bt}}{2}\sqrt{\frac{S_c}{\pi t}}\right] \qquad (32)$$

4 NUSSELT NUMBER

The non-dimensional form of Nusselt number is given by:

$$N_u = -\left(\frac{\partial \theta(y,t)}{\partial y}\right)_{y=0} = -\frac{1}{2\sqrt{t}}\left(\frac{\partial \theta(y,t)}{\partial \eta}\right)_{\eta=0}, \tag{33}$$

Hence,

$$N_u = \left(t\sqrt{R} + \frac{\sqrt{t}P_r}{2\sqrt{R}}\right) erf\left(\sqrt{\frac{Rt}{P_r}}\right) + \frac{\sqrt{t}P_r}{\sqrt{\pi}} e^{-\frac{Rt}{P_r}}. \tag{34}$$

5 SHERWOOD NUMBER

The Sherwood number is given by:

$$S_h = -\left(\frac{\partial C(y,t)}{\partial y}\right)_{y=0} = -\frac{1}{2\sqrt{t}}\left(\frac{\partial C(y,t)}{\partial \eta}\right)_{\eta=0}, \tag{35}$$

Thus,

$$S_h = \sqrt{\frac{tS_c}{\pi}}\left(\sqrt{S_c} - 1\right). \tag{36}$$

Result and Discussion

The numerical results were analysed by computing the finite element solution for the velocity and temperature. Different values of the suction parameter, radiation parameter, Grash of number, magnetic parameter and Prandtl number were used in the cases of impulsive (c = 0) and uniformly accelerated (c = 1) movement of the plate at y = 0.The following values of the above parameters were considered: suction parameter S = 1, 3, 5; radiation parameter Rd = 0.1, 1, 10; Grash of number Gr = 1, 5, 10; magnetic parameter H = 2, 4, 6 and Prandtl number Pr = 0.71(for air), 3(for the saturated liquid Freon at 273.3K), 7(for water). The accuracy of the numerical results was verified by comparing the previous results of Rajput and Sahu [13] with the current finite element solution when the parameter S is set to zero. In figures 2 and 3, the velocity and temperature profiles were compared with the available exact solution obtained by Rajput and Sahu. It was observed that the present numerical results are in good agreement with the exact solution.

The effects of the suction parameter S on the time development of the velocity and temperature of the fluid at the centre of the channel (y=0.5) are shown in figures 4-6. It was observed that both the velocity and the temperature of the fluid decrease with increasing suction parameter. The suction and injection through the plates transfer the fluid near the stationary plate (which has lower velocity) to the centre of the channel (which has higher velocity). This causes the flow velocity at the centre of the channel to decrease. Since the fluid near the stationary plate has a lower temperature than the fluid at the centre of the channel, the fluid transfer due to suction and injection results in a decrease in fluid temperature at the centre of the channel.

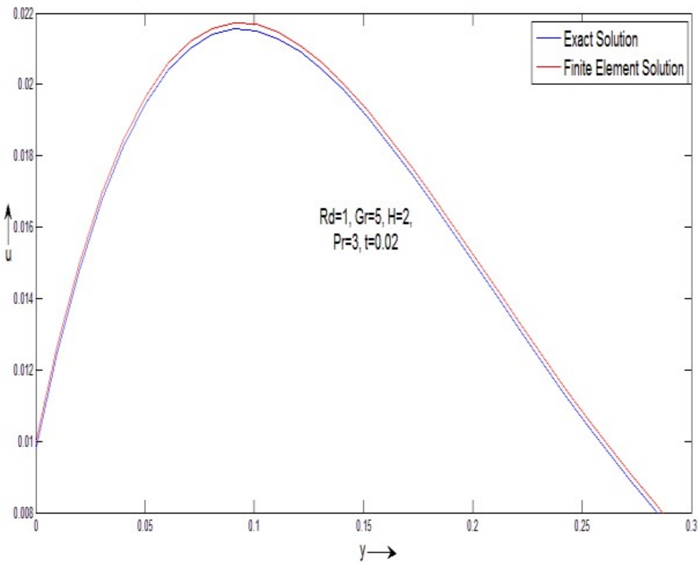


Figure 2: Comparison of Velocity Profiles in the Case of Uniformly Accelerated Movement of the Plate

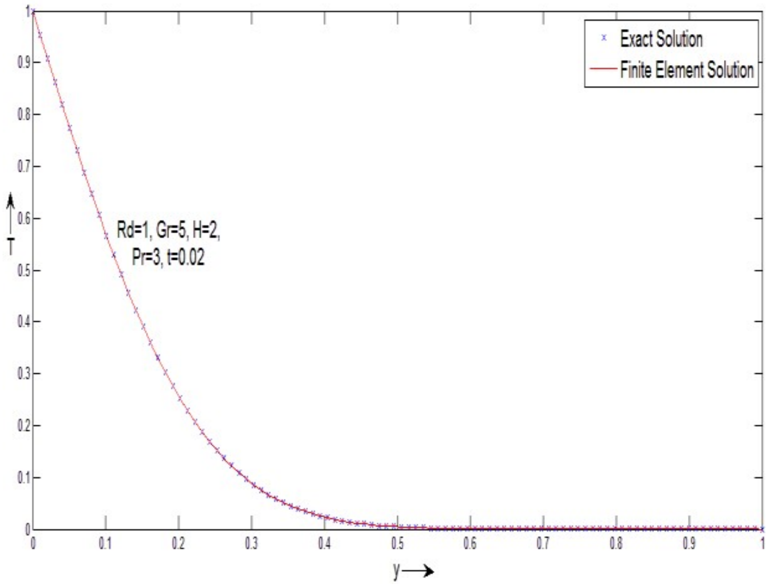


Figure 3: Comparison of Temperature Profiles

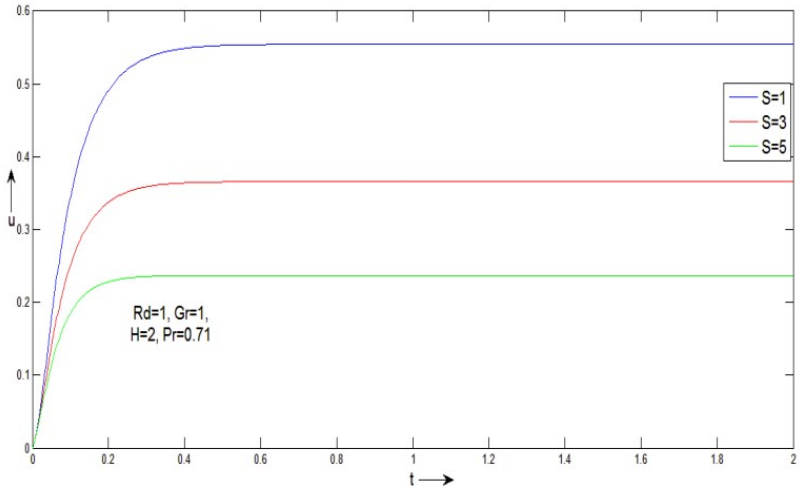


Figure 4: Time Development of Velocity for Different Values of S in the Case of Impulsive Movement of the Plate

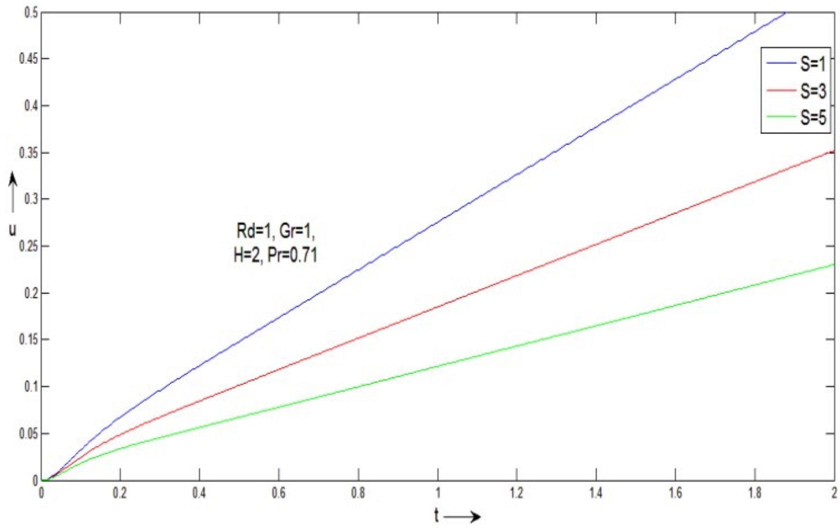


Figure 5: Time Development of Velocity for Different Values of S in the Case of Uniformly Accelerated Movement of the Plate

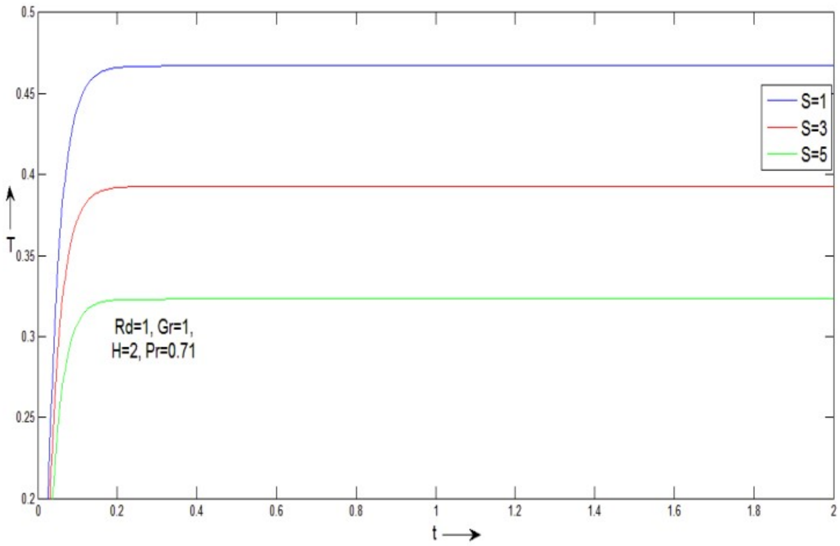
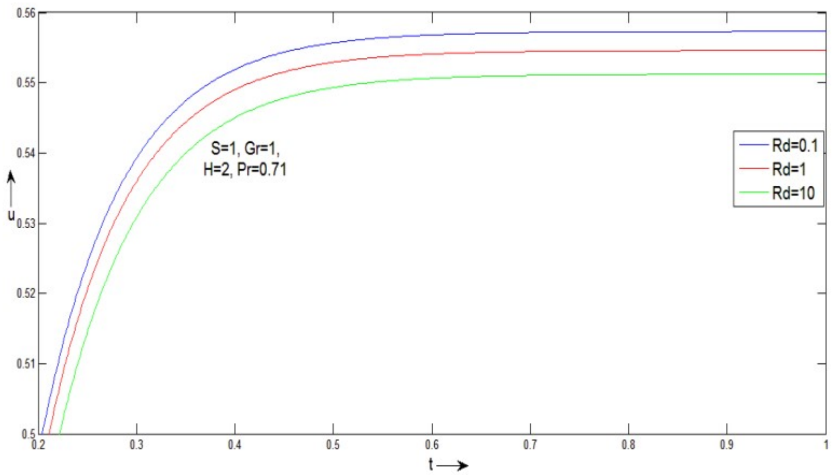


Figure 6: Time Development of Temperature for Different Values of S



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