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# Non-Perturbative Approximate Solution of Fractional Riccati Type Equation

Prabhat Kumar[1,a], and M.M. Dixit[1, b]

[1] *Department of Applied Sciences & Humanities, NERIST, Nirjuli, Department of Mathematics, A.P. India*
[2] *Department of Applied Sciences & Humanities, NERIST, Nirjuli, Department of Mathematics, A.P. India*

[a]Corresponding author: prabhats1718@gmail.com
[b] mmdixit79@gmail.com

**Abstract.** In the present paper a non-perturbative approximate analytic solution is derived for the fractional Riccati type equation by using Adomian Decomposition Method (ADM). The decomposition series solution is very rapidly convergent, and only a few terms of the series solution leads to a very good approximation with the actual solution of the problem. The present method performs extremely well in terms of accuracy, efficiency and simplicity.

**Keywords:** ADM, Power Series Method, Riccati equation.

## INTRODUCTION

Recently a great deal of interest has been focused on Adomian's Decomposition Method (ADM) and its applications to wide class of physical problems containing fractional derivatives [5,6,11,12,13]. The decomposition method employed here is adequately discussed in the published literature [3,4,16], but it still deserves emphasis to point out the very significant advantages over other methods. The said method can also be an effective procedure for the solution of fractional Riccati type equation.

The fractional differential equations have been used to model problems in Physics [2,9], Fluid Mechanics [17,18] and wave propagation phenomena [7,8]. In mathematics, a Riccati equation is any first order ordinary differential equation that is quadratic in the unknown function. In other words, it is an equation of the form

$$\frac{dy(x)}{dx} = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x),$$

where $q_0(x) \neq 0$ and $q_2(x) \neq 0$.

If $q_0(x) \neq 0$, the equation reduces to a Bernoulli equation, while if $q_2(x) = 0$, the equation becomes a first order linear ordinary differential equation. The equation is named after Count Jacopo Francesco Riccati (1676- 1754). More generally, the term "Riccati equation" is used to refer to matrix equations with an analogous quadratic term, which occur in both continuous-time and discrete-time linear quadratic-Gaussian control. The steady-state (non dynamic) version of these is referred to as the algebraic Riccati equation.

The Riccati type equation is one of the basic equations in theoretical physics and has been the focus of many studies. In the present paper we implemented the ADM to the fractional Riccati type equation which is given by

$$\frac{d^{1/2}y(x)}{dx^{1/2}} = -xy(x) + xy^2(x), \qquad (1)$$

where $\frac{d^{1/2}}{dx^{1/2}}$ is the fractional differential operator of order $1/2$ [1,19]. In these schemes the solution constructed in power series with easily computable components.

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## MATHEMATICAL ASPECTS OF FRACTIONAL CALCULUS

Many definitions of fractional calculus are used to solve the problems of fractional differential equations. The most frequently encountered definitions include Riemann- Liouville, Caputo, Weyl and Riesz fractional operator. We introduce the following definitions [1,2].

### DEFINITION

Let $\alpha \notin R^{+}$ .The integral operator $I^{\alpha}$ defined on the usual Lebesgue space L (a, b) by

$$I^{\alpha} f(x) = \frac{d^{-\alpha}}{dx^{-\alpha}} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} f(t)\,dt, \tag{2}$$

for $a \le x \le b$ ,is called Riemann-Liouville fractional integral operator of order $\alpha > 0$ .

### DEFINITION

The Riemann-Liouville definition of fractional order derivative is

$$D^{\alpha} f(x) = \frac{d^{\alpha}}{dx^{\alpha}} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^{m}}{dx^{m}} \int_{0}^{x} (x-t)^{m-\alpha-1} f(t)\,dt, \tag{3}$$

where $m$ is an integer that satisfies $m-1 \le \alpha < m$.

### DEFINITION

A modified fractional differential operator $D^{\alpha}$ proposed by Caputo is given by

$$D^{\alpha} f(x) = \frac{d^{\alpha}}{dx^{\alpha}} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha-1} f^{m}(t)\,dt, \tag{4}$$

Where $\alpha \notin R^{+}$, is the order of operation and $m$ is an integer that satisfies $m-1 \le \alpha < m$.

### ANALYSIS OF THE METHOD

We consider the fractional Riccati type differential eq. (1), where $\frac{d^{1/2}}{dx^{1/2}}$ is the Riemann-Liouville differential operator of order $1/2$. In the standard form used in the ADM, the eq. (1) is rewritten as

$$D^{1/2} y(x) + xy(x) - xNy(x) = 0, \tag{5}$$

where the operator $D^{1/2} \equiv \frac{d^{1/2}}{dx^{1/2}}$ and the nonlinear function $Ny(x) = y^{2}(x)$.

Now, we have

$$D^{1/2} y(x) = -xy(x) + xNy(x), \tag{6}$$

Operating with the integral operator $\frac{d^{-1/2}}{dx^{-1/2}}$, we have

$$y(x)=\frac{c}{\sqrt{x}}-\frac{d^{-1/2}}{dx^{-1/2}}\{xy(x)\}+\frac{d^{-1/2}}{dx^{-1/2}}\{xNy(x)\} \qquad (7)$$

where $c$ is an arbitrary constant.

Following the analysis of ADM [3,4], we expect the decomposition of the solution into a sum of components to be defined by the decomposition series

$$y(x)=\sum_{n=0}^{\infty} y_n(x) \qquad (8)$$

and the nonlinear term $Ny(x)$ is replaced by $\sum_{n=0}^{\infty} A_n$ , where $A_n$ are a set of Adomian's special polynomials [11,19]. Hence, we can write

$$\sum_{n=0}^{\infty} y_n(x)=\frac{c}{\sqrt{x}}-D^{-1/2}\left\{x\sum_{n=0}^{\infty} y_n(x)\right\}+D^{-1/2}\left\{x\sum_{n=0}^{\infty} A_n\right\} \qquad (9)$$

Now assuming the nonlinear function $Ny(x)$ is analytic, these $A_n$ polynomials can be calculated for all forms of non-linearity according to specific algorithms constructed in [3,4,10]. These $A_n$ polynomials depend, of course, on the particular non-linearity. For the present problem, we have [3,4]

$$\sum_{n=0}^{\infty} A_n = Ny(x)=y^2(x) \qquad (10)$$

It follows that

$$A_0 = y_0^{\,2}(x)$$
$$A_1 = 2y_0(x)y_1(x)$$
$$A_2 = y_1^{\,2}(x)+2y_0(x)y_2(x)$$
$$A_3 = 2y_1(x)y_2(x)+2y_0(x)y_3(x)$$

and so on. (11)

Using the $A_n$ polynomials and identifying the zero component $y_0(x)$ by $\frac{c}{\sqrt{x}}$, the remaining components, where $n \geq 0$, can be determined by using the following recurrence relation [3,4]:

$$y_{n+1}(x)=-D^{-1/2}\{xy_n(x)\}+D^{-1/2}\{xA_n\},\ n\geq 0 . \qquad (12)$$

Consequently, we find that

$$y_0(x)=\frac{c}{\sqrt{x}},$$

$$y_1(x)=\frac{2c^2}{\sqrt{\pi}}x^{1/2}-\frac{c\sqrt{\pi}}{2}x,$$

$$y_2(x)=\frac{16c^3}{3\pi}x^{3/2}-\left(\frac{3\pi}{8}+\frac{3}{4}\right)c^2x^2+\frac{8c}{15}x^{5/2},$$

$$y_3(x)=\frac{704c^4}{45\pi\sqrt{\pi}}x^{5/2}-\left(\frac{15\pi\sqrt{\pi}}{64}+\frac{35\sqrt{\pi}}{32}+\frac{5}{3\sqrt{\pi}}\right)c^3x^3+\left(\frac{872}{525\sqrt{\pi}}+\frac{4\sqrt{\pi}}{7}\right)c^2x^{7/2}-\frac{7\sqrt{\pi}}{48}cx^4,$$

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$$y_4(x)=\frac{75776}{1575\pi^2}c^2x^{7/2}-\left(\frac{525\pi^2}{4096}+\frac{2065\pi}{2048}+\frac{1225}{384}+\frac{77}{18\pi}\right)c^4x^4+$$
$$\left(\frac{52\pi}{105}+\frac{1784}{735}+\frac{957184}{165375\pi}\right)c^3x^{9/2}-\left(\frac{3519\pi}{10240}+\frac{327}{800}\right)c^2x^5+\frac{32}{297}cx^{11/2}$$

and so on. (13)

Therefore, all components of $y(x)$ are calculable and from (8), the solution is completely determined. The expression $\phi_n(x)=\sum_{i=0}^{n-1}y_i(x)$ is the $n-$ term approximation to $y(x)$. Here it is to be noted that, the decomposition series solution is very rapidly convergent [4, 5, 6, 13], and only a few terms of the series solution leads to a very good approximation with the actual solution of the problem. Generally, only a few terms are sufficient for most purpose and we can proceed further with little effort [5,6,10,13]. We consider $\phi_5(x)$, i.e., five-term approximation, the solution is given by

$$y(x)\approx\phi_5(x)=y_0(x)+y_1(x)+y_2(x)+y_3(x)+y_4(x)$$
$$=\frac{c}{\sqrt{\pi}}+\frac{2c^2}{\sqrt{\pi}}x^{1/2}-\frac{c\sqrt{\pi}}{2}x+\frac{16c^3}{3\pi}x^{3/2}-\left(\frac{3\pi}{8}+\frac{3}{4}\right)c^2x^2+$$
$$\left(\frac{8c}{15}+\frac{704c^4}{45\pi\sqrt{\pi}}\right)x^{5/2}-\left(\frac{15\pi\sqrt{\pi}}{64}+\frac{35\sqrt{\pi}}{32}+\frac{5}{3\sqrt{\pi}}\right)c^3x^3+$$
$$\left\{\left(\frac{872}{525\sqrt{\pi}}+\frac{4\sqrt{\pi}}{7}\right)c^2+\frac{75776}{1575\pi^2}c^5\right\}x^{7/2}-$$
$$\left\{\frac{7\sqrt{\pi}}{48}c+\left(\frac{525\pi^2}{4096}+\frac{2065\pi}{2048}+\frac{1225}{384}+\frac{77}{18\pi}\right)c^4\right\}x^4+$$
$$\left(\frac{52\pi}{105}+\frac{1784}{735}+\frac{957184}{165375\pi}\right)c^3x^{9/2}-\left(\frac{3519\pi}{10240}+\frac{327}{800}\right)c^2x^5+\frac{32}{297}cx^{11/2} \quad (14)$$

## VERIFICATION BY THE POWER SERIES METHOD

In view of the fractional differential eq. (1), we can take the solution in the form of the following fractional power series

$$y(x)=\frac{c}{\sqrt{x}}+\sum_{n=0}^{\infty}a_nx^{n/2}. \quad (15)$$

Substituting this expansion (15) into eq. (1) we obtain

$$\sum_{n=0}^{\infty}a_n\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}x^{(n-1)/2}+x\left\{\frac{c}{\sqrt{x}}+\sum_{n=0}^{\infty}a_nx^{n/2}\right\}-x\left\{\frac{c}{\sqrt{x}}+\sum_{n=0}^{\infty}a_nx^{n/2}\right\}^2=0. \quad (16)$$

By equating the co-efficient of different powers of $x$, we obtain

$a_0=0,$

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$$a_1 = \frac{2c^2}{\sqrt{\pi}},$$

$$a_2 = -\frac{c\sqrt{\pi}}{2},$$

$$a_3 = \frac{16c^3}{3\pi},$$

$$a_4 = -\left(\frac{3\pi}{8} + \frac{3}{4}\right)c^2,$$

$$a_5 = \frac{8c}{15} + \frac{704c^4}{45\pi\sqrt{\pi}},$$

$$a_6 = -\left(\frac{15\pi\sqrt{\pi}}{64} + \frac{35\sqrt{\pi}}{32} + \frac{5}{3\sqrt{\pi}}\right)c^3,$$

$$a_7 = \left(\frac{4\sqrt{\pi}}{7} + \frac{872}{525\sqrt{\pi}}\right)c^2 + \frac{75776}{1575\pi^2}c^5,$$

$$a_8 = -\left\{\frac{7\sqrt{\pi}}{48}c + \left(\frac{525\pi^2}{4096} + \frac{2065\pi}{2048} + \frac{1225}{384} + \frac{77}{18\pi}\right)c^4\right\},$$

$$a_9 = \left\{\frac{52\pi}{105} + \frac{1784}{735} + \frac{957184}{165375\pi}\right\}c^3 + \frac{8388608}{55125\pi^2}c^6,$$

$$a_{10} = -\left\{\left(\frac{3519\pi}{10240} + \frac{327}{800}\right)c^2 + \left(\frac{33075\pi^2\sqrt{\pi}}{524288} + \frac{190575\pi\sqrt{\pi}}{262144} + \frac{178479\sqrt{\pi}}{49152} + \frac{12243}{1280\sqrt{\pi}} + \frac{296}{25\pi\sqrt{\pi}}\right)c^5\right\}$$

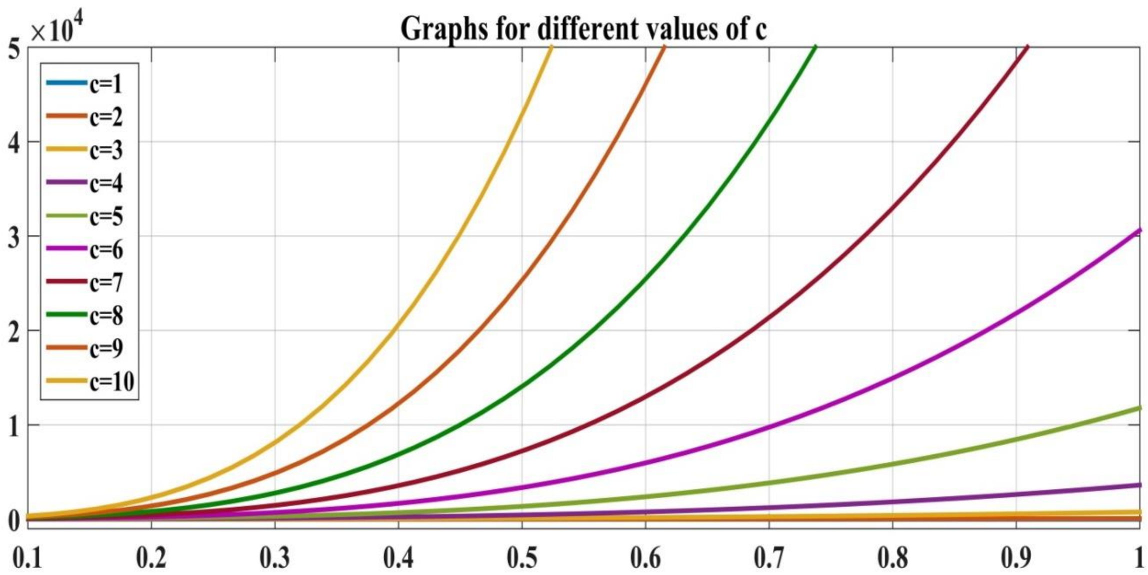
and so on. (17)

The rest of the terms can be calculated in a similar manner. Therefore, the solution of eq. (1) is

$$\begin{aligned} y(x) = {} & \frac{c}{\sqrt{\pi}} + \frac{2c^2}{\sqrt{\pi}}x^{1/2} - \frac{c\sqrt{\pi}}{2}x + \frac{16c^3}{3\pi}x^{3/2} - \left(\frac{3\pi}{8} + \frac{3}{4}\right)c^2x^2 + \\ & \left(\frac{8c}{15} + \frac{704c^4}{45\pi\sqrt{\pi}}\right)x^{5/2} - \left(\frac{15\pi\sqrt{\pi}}{64} + \frac{35\sqrt{\pi}}{32} + \frac{5}{3\sqrt{\pi}}\right)c^3x^3 + \\ & \left\{\left(\frac{872}{525\sqrt{\pi}} + \frac{4\sqrt{\pi}}{7}\right)c^2 + \frac{75776}{1575\pi^2}c^5\right\}x^{7/2} - \\ & \left\{\frac{7\sqrt{\pi}}{48}c + \left(\frac{525\pi^2}{4096} + \frac{2065\pi}{2048} + \frac{1225}{384} + \frac{77}{18\pi}\right)c^4\right\}x^4 + \ldots, \end{aligned} \tag{18}$$

Retaining up to the terms of the order $x^4$ .The series (18) is in complete agreement with that of (14). It follows immediately that the accuracy of the solution can be improved by computing more terms in the decomposition method. This is due to the rapid convergences of the decomposition solution [4,5,6,13]. The above results may be compared obtained in [20].




## CONCLUSION

The present problem deals with the solution of Riccati type fractional differential equation with the help of ADM. The solution obtained by this method is verified with that of Power Series Method and the solutions obtained by both these methods are found to be in complete agreement justifying the concept of ADM. The advantages of this global methodology lies in the fact that it not only leads to an analytical continuous approximation which is very rapidly convergent [4, 5, 6, 13] but also shows the dependence, giving insight into the character and behavior of the solution just as in a closed form solution [4, 6, 11].

The present analysis exhibits the applicability of the decomposition method to solve a non-linear fractional differential equation. Furthermore, this method does not require any transformation techniques, linearization or discretization of the variables and it does not make closure approximation or smallness assumption. This technique may be applied to the nonlinear partial differential equations such as KdV (Korteweg-De Vries) equation, nonlinear Schrödinger equation and Barger's equation.

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