e-ISSN 2320 –7876 www.ijfans.org
Vol.6, Iss.4, Jul 2017

AN APPRAISAL OF THE LITERATURE AND A NEW STRATAGEM ON FUZZY INNER PRODUCT SPACE (FIPS)

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Abstract

We attempt a rigorous definition of FIPS in this study. We have focused our attention on a few techniques that had already been discussed in the literature. It has become necessary to properly analyse the existing literature on FIPS to guide future research in light of various new studies. We looked at an alternative method of approaching the concept of the FIPS, beginning with the work of P. Majumdar and S.K. Samanta. As we honed their definition, we were able to zero in on many peculiarities of the FIPS. Finally, we demonstrated that this FIPS yields anNădăban-Dzitac fuzzy norm. A few difficulties are then presented as a conclusion.

Index Terms: Fuzzy Norm**,** FIPS**,** Fuzzy Hilbert Space

1. Overview

The foundation for fuzzy systematic study was created by the work of A.K. Katsaras [1, 2]. Mathematical minds are pondering the implications of this idea. C. Felbin [3] brought in the scheme of a fuzzy standard in regular grid in 1992 by providing an authentic fuzzy set to each space component.

To conduct functional analysis, we must comprehend Hilbert's space. They play a crucial role in practically every discipline of mathematics and science, including, but not restricted to, differential equation analysis, quantum logic, quantum physics, Fourier analysis, and quantum computing. Several mathematicians have tried to define the FIPS, with varied degrees of success.

The concept of a FIPS has been the theme of very few studies, notwithstanding the innumerable papers written on fuzzy standards and their wide range of applications.

Despite the low probability that any specific outcome will be significant, we will treat them all as equally important. In addition, there is a precise meaning of the FIPS, the discovery of which would lead to a global comprehension of the problem and a vast array of commercial applications.

When we set out to write this article, we did so to contribute to the body of knowledge. As essential to us is drawing attention to seminal work done by others.

Two years after refining R. Biswas' concept of the inner product space, FIPS was first proposed by Kumar. They proved that this expression is only applicable to normal linear spaces and that characterising a FIPS in regards of the inverse of a vector is difficult. Fuzzy co-norm functions were also incorporated into their research.

The author Shin built and characterised “fuzzy semi-inner product space” and looked into its assets in [3]. These standards are not confined to linear dimensions that are either physical or symbolic.

2008 marked the start of S.K. Samanta and P. Majumdar's [4] quest for a convincing description of a FIPS. They say it's difficult to give a clear, up-to-date description of the product's hazy core. This assumption must be legally separated because the conventional ratio Cauchy cannot be acquired from earlier assumptions (FIP).

Many convergence theorems, notably the Cauchy-Schwartz inequality and the Pythagorean Theorem, were developed in this context.

This notion is confined to linear domains over R, a fairly restrictive context. It's also tough to use real fuzzy numbers in practise, so there's room for development there as well.

In 2018, [1] created two unique theoretical models for FIPS and examined a number of their fundamental aspects. This piece is structured as follows: A survey of the relevant literature is presented in Part II. This method is helpful since it would provide greater clarity for the reader regarding developing ideas related to the fuzzy inner product space. For mathematicians interested in learning more, this section is a great place to get in. Section 3 suggests a new

e-ISSN 2320 –7876 www.ijfans.org
Vol.6, Iss.4, Jul 2017

description of the FIPS based on the research of Samanta and Majumdar[4]. We enhanced the FIPS function and modified the inner product space idea proposed by P. Majumdar and S.K. Samanta. The essay's final part addresses specific open questions and reaches some conclusions.

2. Groundwork

Definition 1. *[10] Let X be a vector space over a field $\mathbb{K}$ and * be a continuous t-norm. A fuzzy set N in $X \times [0, \infty]$ is called a fuzzy norm on X if it satisfies:*

(N1) $N(x, 0) = 0, (\forall)x \in X$;

(N2) $[N(x, t) = 1, (\forall)t > 0]$ *iff* $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall)\, x \in X, (\forall)\, t \geq 0, (\forall)\, \lambda \in \mathbb{K}^*$;

(N4) $N(x + y, t + s) \geq N(x, t) * N(y, t), (\forall)x, y \in X, (\forall)t, s \geq 0$;

(N5) $(\forall)x \in X, N(x, \cdot)$ *is left continuous and* $\lim_{t\to\infty} N(x, t) = 1$.

The triplet $(X, N, *)$ will be called fuzzy normed linear space (briefly FNLS).

Definition 2. *[14] A fuzzy inner product space (FIP-space) is a pair (X, P), where X is a linear space over $\mathbb{C}$ and P is a fuzzy set in $X \times X \times \mathbb{C}$ s.t.*

(FIP1) *For* $s, t \in \mathbb{C},\ P(x + y, z, |t| + |s|) \geq \min\{P(x, z, |t|), P(y, z, |s|)\}$;

(FIP2) *For* $s, t \in \mathbb{C},\ P(x, y, |st|) \geq \min\left\{P\left(x, x, |s|^2\right), P\left(y, y, |t|^2\right)\right\}$;

(FIP3) *For* $t \in \mathbb{C},\ P(x, y, t) = P\left(y, x, \bar{t}\right)$;

(FIP4) $P(\alpha x, y, t) = P\left(x, y, \frac{t}{|\alpha|}\right), t \in \mathbb{C}, \alpha \in \mathbb{C}^*$;

(FIP5) $P(x, x, t) = 0, (\forall)\, t \in \mathbb{C} \setminus \mathbb{R}^+$;

(FIP6) $[P(x, x, t) = 1, (\forall)t > 0]$ *iff* $x = 0$;

(FIP7) $P(x, x, \cdot) : \mathbb{R} \to [0, 1]$ *is a monotonic non-decreasing function of $\mathbb{R}$ and* $\lim_{t\to\infty} P(x, x, t) = 1$.

P will be called the fuzzy inner product on X.

Definition 3. *[15] A fuzzy inner product space (FIP-space) is a triplet $(X; P; *)$, where X is a real linear space, $*$ is a continuous t-norm and P is a fuzzy set on $X^2 \times \mathbb{R}$ satisfying the following conditions for every $x; y; z \in X$ and $t \in \mathbb{R}$.*

(FIP1) $P(x, y, 0) = 0$;

(FIP2) $P(x, y, t) = P(y, x, t)$;

(FIP3) $P(x, x, t) = H(t), \forall t \in \mathbb{R}$ *iff* $x = 0$, *where* $H(t) = \begin{cases} 1, & if\ t > 0 \\ 0, & if\ t \leq 0 \end{cases}$;

(FIP4) *For any real number* α, $P(\alpha x, y, t) = \begin{cases} P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0 \\ 1 - P\left(x, y, \frac{t}{-\alpha}\right), & if\ \alpha < 0 \end{cases}$;

(FIP5) $\sup_{s+r=t} (P(x, z, s) * P(y, z, r)) = P(x + y, z, t)$;

(FIP6) $P(x, y, \cdot) : \mathbb{R} \to [0, 1]$ *is continuous on* $\mathbb{R} \setminus \{0\}$;

(FIP7) $\lim_{t\to\infty} P(x, y, t) = 1$.

Definition 4. *[16] A fuzzy inner product space (FIP - space) is a triplet $(X, P, *)$, where X is a real linear space, $*$ is a continuous t-norm and P is a fuzzy set in $X \times X \times \mathbb{R}$ s.t. the following conditions hold for every $x, y, z \in X$ and $s, t, r \in \mathbb{R}$.*

(FI-1) $P(x, x, 0) = 0$ *and* $P(x, x, t) > 0,\ (\forall)t > 0$;

(FI-2) $P(x, x, t) \neq H(t)$ *for same* $t \in \mathbb{R}$ *iff* $x \neq 0$;

(FI-3) $P(x, y, t) = P(y, x, t)$;

(FI-4) *For any real number* α, $P(\alpha x, y, t) = \begin{cases} P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P\left(x, y, \frac{t}{-\alpha}\right), & if\ \alpha < 0 \end{cases}$

(FI-5) $\sup_{s+r=t} (P(x, z, s) * P(y, z, r)) = P(x + y, z, t)$;

(FI-6) $P(x, y, \cdot) : \mathbb{R} \to [0, 1]$ *is continuous on* $\mathbb{R} \setminus \{0\}$;

(FI-7) $\lim_{t \to \infty} P(x, y, t) = 1$.

Definition 5. *[14] Let X be a linear space over $\mathbb{R}$. A fuzzy set P in $X \times X \times \mathbb{R}$ is called fuzzy real inner product on X if $(\forall)x, y, z \in X$ and $t \in \mathbb{R}$, the following conditions hold:*

(FI-1) $P(x, x, 0) = 0,\ (\forall)t < 0$;

(FI-2) $[P(x, x, t) = 1,\ (\forall)t > 0]\ iff\ x = 0$;

(FI-3) $P(x, y, t) = P(y, x, t)$;

(FI-4) $P(\alpha x, y, t) = \begin{cases} P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha < 0 \end{cases}$

(FI-5) $P(x + y, z, t + s) \geq \min \{P(x, z, t), P(y, z, s)\}$;

(FI-6) $\lim_{t \to \infty} P(x, y, t) = 1$.

The pair (X, P) is called fuzzy real inner space.

Definition 6. *[28] A fuzzy set in $\mathbb{R}$, namely a mapping $x : \mathbb{R} \to [0, 1]$, with the following properties:*

(1) *x is convex, that is,* $x(t) \geq \min \{x(s), x(r)\}$ *for* $s \leq t \leq r$;

(2) *x is normal, that is,* $(\exists)\, t_0 \in \mathbb{R} : x(t_0) = 1$;

(3) *x is upper semicontinuous, that is,* $(\forall)t \in \mathbb{R}, (\forall)\alpha \in [0, 1] : x(t) < \alpha, (\exists)\delta > 0$ *s.t.* $|s - t| < \delta \Rightarrow x(s) < \alpha$

is called fuzzy real number. We denote by $\mathfrak{F}(\mathbb{R})$ the set of all fuzzy real numbers.

Definition 7. *[29] The arithmetic operation $+, -, \cdot, /$ on $\mathfrak{F}(\mathbb{R})$ are defined by:*

$$(x + y)(t) = \bigvee_{s \in \mathbb{R}} \min \{x(s), y(t - s)\},\ (\forall)\, t \in \mathbb{R};$$

$$(x - y)(t) = \bigvee_{s \in \mathbb{R}} \min \{x(s), y(s - t)\},\ (\forall)\, t \in \mathbb{R};$$

$$(xy)(t) = \bigvee_{s \in \mathbb{R}^*} \min \{x(s), y(t/s)\},\ (\forall)\, t \in \mathbb{R};$$

$$(x/y)(t) = \bigvee_{s \in \mathbb{R}} \min \{x(ts), y(s)\},\ (\forall)\, t \in \mathbb{R}.$$

Remark 1. *Let* $x \in \mathfrak{F}(\mathbb{R})$ *and* $\alpha \in (0,1]$. *The α-level sets* $[x]_\alpha = \{t \in \mathbb{R} : x(t) \geq \alpha\}$ *are closed intervals* $\left[x_\alpha^-, x_\alpha^+\right]$.

Definition 8. *[18] Let X be a linear space over* $\mathbb{R}$. *A fuzzy inner product on X is a mapping* $< \cdot, \cdot >: X \times X \to \mathfrak{F}(\mathbb{R})$ *s.t.* $(\forall)x, y, z \in X, (\forall)r \in \mathbb{R}$, *we have:*

(IP1) $< x + y, z > = < x, z > \oplus < y, z >$;

(IP2) $< rx, y > = \tilde{r} < x, y >$, *where* $\tilde{r} = \begin{cases} 1, & if\ t = r \\ 0, & if\ t \neq r \end{cases}$;

(IP3) $< x, y > = < y, x >$;

(IP4) $< x, x > \geq 0$;

(IP5) $\inf_{\alpha \in (0,1]} < x, x >_\alpha^- = 0$ if $x \neq 0$;

(IP6) $< x, x > = \tilde{0}\ iff\ x = 0$.

The pair $(X, \langle \cdot, \cdot \rangle)$ *is called fuzzy inner product space.*

Definition 9. *[25] A fuzzy inner product space is a triplet* $(X, P, *)$, *where X is a fuzzy set in* $X \times X \times \mathbb{R}$ *satisfying the following conditions for every* $x, y, z \in X$ *and* $t, s \in \mathbb{R}$:

(FI1) $P(x, y, 0) = 0$;

(FI2) $P(x, y, t) = P(y, x, t)$;

(FI3) $[P(x, x, t) = 1, (\forall)t > 0]\ iff\ x = 0$;

(FI4) $$(\forall)\, \alpha \in \mathbb{R}, t \neq 0, \quad P(\alpha x, y, t) = \begin{cases} P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0 \\ 1 - P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha < 0 \end{cases};$$

(FI5) $P(x, z, t) * P(y, z, s) \leq P(x + y, z, t + s), (\forall)t, s > 0$;

(FI6) $\lim_{t \to \infty} P(x, y, t) = 1$.

Definition 10. *[26] Let X be a linear space over* $\mathbb{R}$. *A fuzzy inner product on X is a mapping* $< \cdot, \cdot >: X \times X \to \mathfrak{F}^*(\mathbb{R})$, *where* $\mathfrak{F}^*(\mathbb{R}) = \{\eta \in \mathfrak{F}(\mathbb{R}) : \eta(t) = 0 \text{ if } t < 0\}$, *with the following properties* $(\forall)x, y, z \in X, (\forall)r \in \mathbb{R}$:

(FIP1) $< x + y, z > = < x, z > \oplus < y, z >$;

(FIP2) $< rx, y > = |\tilde{r}| < x, y >$;

(FIP3) $< x, y > = < y, x >$;

(FIP4) $x \neq 0 \Rightarrow < x, x > (t) = 0, (\forall)t < 0$;

(FIP5) $< x, x > = \tilde{0}\ iff\ x = 0$.

The pair $(X, < \cdot, \cdot >)$ *is called fuzzy inner product space.*

3. Introducing a Novel Method for the Study of FIPS

e-ISSN 2320 –7876 www.ijfans.org
Vol.6, Iss.4, Jul 2017

Definition 11. *Let H be a linear space over $\mathbb{C}$. A fuzzy set P in $H \times H \times \mathbb{C}$ is called a fuzzy inner product on H if it satisfies:*

(FIP1) $P(x, x, v) = 0, (\forall)\, x \in H, (\forall)\, v \in \mathbb{C} \setminus \mathbb{R}_+^*$;

(FIP2) $P(x, x, t) = 1, (\forall)\, t \in \mathbb{R}_+^*$ *if and only if* $x = 0$;

(FIP3) $P(\alpha x, y, v) = P\left(x, y, \frac{v}{|\alpha|}\right), (\forall)\, x, y \in H, (\forall)\, v \in \mathbb{C}, (\forall)\, \alpha \in \mathbb{C}^*$;

(FIP4) $P(x, y, v) = P(y, x, \bar{v}), (\forall)\, x, y \in H, (\forall)\, v \in \mathbb{C}$;

(FIP5) $P(x + y, z, v + w) \geq \min\{P(x, z, v), P(y, z, w)\}, (\forall)\, x, y, z \in H, (\forall)\, v, w \in \mathbb{C}$;

(FIP6) $P(x, x, \cdot) : \mathbb{R}_+ \to [0, 1], (\forall)\, x \in H$ *is left continuous and* $\lim_{t \to \infty} P(x, x, t) = 1$;

(FIP7) $P(x, y, st) \geq \min\left\{P\left(x, x, s^2\right), P\left(y, y, t^2\right)\right\}, (\forall)\, x, y \in H, (\forall)\, s, t \in \mathbb{R}_+^*$.

The pair (H, P) will be called fuzzy inner product space.

Example 1. *Let H be a linear space over $\mathbb{C}$ and $<\cdot,\cdot>: H \times H \to \mathbb{C}$ be an inner product. Then* $P : H \times H \times \mathbb{C} \to [0, 1]$,

$$P(x, y, s) = \begin{cases} \frac{s}{s + |<x,y>|}, & \text{if } s \in \mathbb{R}_+^* \\ 0, & \text{if } s \in \mathbb{C} \setminus \mathbb{R}_+^* \end{cases}$$

is a fuzzy inner product on H.

Let verify now the conditions from the definition.

(FIP1) $P(x, x, v) = 0, (\forall)\, x \in H, (\forall)\, v \in \mathbb{C} \setminus \mathbb{R}_+^*$ is is obvious from definition of *P*.

(FIP2) $P(x, x, t) = 1, (\forall)\, v \in \mathbb{R}_+^* \Leftrightarrow t + |<x, x>| = t, (\forall)\, t > 0 \Leftrightarrow |<x, x>| = 0 \Leftrightarrow x = 0$.

(FIP3) $P(\alpha x, y, v) = P\left(x, y, \frac{v}{|\alpha|}\right), (\forall)\, x, y \in H, (\forall)\, v \in \mathbb{C}, (\forall)\ \alpha \in \mathbb{C}$ is obvious for $v \in \mathbb{C} \setminus \mathbb{R}_+^*$.
If $v \in \mathbb{R}_+^*$, than

$$P(\alpha x, y, v) = \frac{v}{v + |<\alpha x, y>|} = \frac{v}{v + |\alpha| \cdot |<x, y>|} = \frac{\frac{v}{|\alpha|}}{\frac{v}{|\alpha|} + |<x, y>|} = P\left(x, y, \frac{v}{|\alpha|}\right).$$

(FIP4) $P(x, y, v) = P(y, x, \bar{v}), (\forall)\, x, y \in H, (\forall)\, v \in \mathbb{C}$ is obvious for $v \in \mathbb{C} \setminus \mathbb{R}_+^*$.
If $v \in \mathbb{R}_+^*$, then $v = \bar{v}$ and

$$P(x, y, v) = \frac{v}{v + |<x, y>|} = \frac{\bar{v}}{\bar{v} + |<y, x>|} = P(y, x, \bar{v})$$

(FIP5) $P\left(x+y, z, v+w\right) \geq \min\left\{P(x, z, v), P(y, z, w)\right\}, (\forall)\, x, y, z \in H, (\forall)\, v, w \in \mathbb{C}$.

If at least one of *v* and *w* is from $\mathbb{C} \setminus \mathbb{R}_{+}^{*}$, then the result is obvious.

If $v, w \in \mathbb{R}_{+}^{*}$, let us assume without loss of generality that $P(x, z, v) \leq P(y, z, w)$. Then

$$\frac{v}{v+|<x,z>|} \leq \frac{w}{w+|<y,z>|} \Rightarrow$$

$$\Rightarrow \frac{v+|<x,z>|}{v} \geq \frac{w+|<y,z>|}{w} \Rightarrow$$

$$\Rightarrow 1+\frac{|<x,z>|}{v} \geq 1+\frac{|<y,z>|}{w} \Rightarrow$$

$$\Rightarrow \frac{|<x,z>|}{v} \geq \frac{|<y,z>|}{w} \Rightarrow$$

$$\Rightarrow \frac{w}{v}\,|<x,z>| \geq |<y,z>| \Rightarrow$$

$$\Rightarrow |<x,z>| + \frac{w}{v}\,|<x,z>| \geq |<x,z>| + |<y,z>| \Rightarrow$$

$$\Rightarrow \frac{v+w}{v}\,|<x,z>| \geq |<x+y,z>| \Rightarrow$$

$$\Rightarrow \frac{|<x,z>|}{v} \geq \frac{|<x+y,z>|}{v+w} \Rightarrow$$

e-ISSN 2320 –7876 www.ijfans.org
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$$\Rightarrow \frac{|<x,z>|}{v}+1 \geq \frac{|<x+y,z>|}{v+w}+1 \Rightarrow$$

$$\Rightarrow \frac{v+|<x,z>|}{v} \geq \frac{(v+w)+|<x+y,z>|}{v+w} \Rightarrow$$

$$\Rightarrow \frac{v}{v+|<x,z>|} \leq \frac{v+w}{(v+w)+|<x+y,z>|} \Rightarrow$$

$$\Rightarrow P(x,z,v) \leq P(x+y,z,v+w)$$

$$\Rightarrow P(x+y,z,v+w) \geq \min\{P(x,z,v),P(y,z,w)\},(\forall)\,x,y,z \in H,(\forall)\,v,w \in \mathbb{C}.$$

(FIP6) $P(x,x,\cdot):\mathbb{R}_+ \to [0,1]$, $(\forall)\, x \in H$ is left continuous function and $\lim_{t\to\infty} P(x,x,t) = 1$.

$$\lim_{t\to\infty} P(x,x,t) = \lim_{t\to\infty} \frac{t}{t+|<x,x>|} = \lim_{t\to\infty} \frac{t}{t\left(1+\frac{|<x,x>|}{t}\right)} = 1.$$

$F(x,x,\cdot)$ is left continuous in $t>0$ follows from definition.

(FIP7) $P(x,y,st) \geq \min\left\{P\left(x,x,s^2\right),P\left(y,y,t^2\right)\right\}$, $(\forall)\, x,y \in H$, $(\forall)\, s,t \in \mathbb{R}_+^*$. If at least one of *s* and *t* is from $\mathbb{C} \setminus \mathbb{R}_+^*$, then the result is obvious.
If $s,t \in \mathbb{R}_+^*$, let us assume without loss of generality that $P\left(x,x,s^2\right) \leq P\left(y,y,t^2\right)$. Then

$$\frac{s^2}{s^2+|<x,x>|} \leq \frac{t^2}{t^2+|<y,y>|} \Leftrightarrow$$

$$t^2\,|<x,x>| \geq s^2\,|<y,y>|.$$

Thus by Cauchy–Schwartz inequality we obtain

$$s\,|<x,y>| \leq \sqrt{|<x,x>|}\cdot s\sqrt{|<y,y>|} \leq \sqrt{|<x,x>|}\cdot t\sqrt{|<x,x>|} = t\,|<x,x>| \Rightarrow$$

$$\Rightarrow s^2\,|<x,y>| \leq st\,|<x,x>| \Rightarrow$$

$$\Rightarrow s^3t+s^2\,|<x,y>| \leq s^3t+st\,|<x,x>| \Rightarrow$$

$$\Rightarrow s^2\,(st+|<x,y>|) \leq st\,(s^2+|<x,x>|) \Rightarrow$$

$$\frac{s^2}{s^2+|<x,x>|} \leq \frac{st}{st+|<x,y>|} \Rightarrow$$

$$\Rightarrow P\left(x,x,s^2\right) \leq P(x,y,st) \Rightarrow$$

$$\Rightarrow P(x,y,st) \geq \min\left\{P\left(x,x,s^2\right), P\left(y,y,t^2\right)\right\}, (\forall)\, x,y \in H, (\forall)\, s,t \in \mathbb{R}_+^*.$$

Proposition 1. *For $x, y \in H$, $v \in \mathbb{C}$ and $\alpha \in \mathbb{C}$ we have*

$$P(x,\alpha y,v) = P\left(x,y,\frac{v}{|\alpha|}\right).$$

Proof. From (FIP3) and (FIP4) it follows $P(x,\alpha y,v) = P(\alpha y,x,\bar{v}) = P\left(y,x,\frac{\bar{v}}{|\alpha|}\right) = P\left(x,y,\frac{\bar{\bar{v}}}{|\alpha|}\right) = P\left(x,y,\frac{v}{|\alpha|}\right)$. □

Proposition 2. *For $x \in H$, $v \in \mathbb{R}_+^*$ we have*

$$P(x,0,v) = 1.$$

Proof. From (FIP3) and (FIP6) it follows

$$P(x,0,v) = P(x,0,2nv) = P(x,x-x,nv+nv) \geq \min\left\{P(x,x,nv), P(x,x,nv)\right\} =$$

$$P(x,x,nv) \xrightarrow{n\to\infty} 1$$

So $P(x,0,v) = 1$. □

Proposition 3. *For $y \in H$, $v \in \mathbb{R}_+^*$ we have*

$$P(0,y,v) = 1.$$

Proposition 4. $P(x,y,\cdot) : \mathbb{R}_+ \to [0,1]$ *is a monotonic non-decreasing function on $\mathbb{R}_+$, $(\forall) x,y \in H$.*

Proof. Let $s,t \in \mathbb{R}_+$, $s \leq t$. Then $(\exists) p$ such that $t = s+p$ and

$$P(x,y,t) = P(x+0,y,s+p) \geq \min\left\{P(x,y,s), P(0,y,p)\right\} = P(x,y,s).$$

Hence $P(x,y,s) \leq P(x,y,t)$ for $s \leq t$. □

Corollary 1. $P(x,y,st) \geq \min\left\{P\left(x,y,s^2\right), P\left(x,y,t^2\right)\right\}$, $(\forall)\, x,y \in H$, $(\forall)\, s,t \in \mathbb{R}_+^*$.

Proof. Let $s,t \in \mathbb{R}_+$, $s \leq t$. Then

$$P\left(x,y,s^2\right) \leq P(x,y,st) \leq P\left(x,y,t^2\right).$$

Hence $P(x,y,st) \geq \min\left\{P\left(x,y,s^2\right), P\left(x,y,t^2\right)\right\}$.

Let now $s,t \in \mathbb{R}_+$, $t \leq s$. Then

$$P\left(x,y,t^2\right) \leq P(x,y,st) \leq P\left(x,y,s^2\right).$$

Hence $P(x,y,st) \geq \min\left\{P\left(x,y,s^2\right), P\left(x,y,t^2\right)\right\}$. □

Proposition 5. $P(x,y,v) \geq \min\left\{P(x,y-z,v), P(x,y+z,v)\right\}$, $(\forall)\, x,y,z \in H$, $(\forall)\, v \in \mathbb{C}$.

Proof.

$$P(x,y,v) = P(x,2y,2v) = P(x,y+z+y-z,v+v) \geq \min\left\{P(x,y+z,v), P(x,y-z,v)\right\}.$$

Hence $P(x,y,v) \geq \min\left\{P(x,y-z,v), P(x,y+z,v)\right\}$. □

Theorem 1. *If (H,P) be a fuzzy inner product space, then $N : X \times [0,\infty) \to [0,1]$ defined by*

$$N(x,t) = P\left(x,x,t^2\right)$$

is a fuzzy norm on X.

Proof.

(N1) $N(x,0) = P(x,x,0) = 0, (\forall)x \in H$ from **(FIP1)**;

(N2) $[N(x,t) = 1, (\forall)t > 0] \Leftrightarrow [P(x,x,t^2) = 1, (\forall)\, t > 0] \Leftrightarrow x = 0$ from **(FIP2)**;

(N3) $N(\lambda x, t) = P(\lambda x, \lambda x, t^2) = P\left(x, \lambda x, \frac{t^2}{|\lambda|}\right) = P\left(\lambda x, x, \overline{\frac{t^2}{|\lambda|}}\right) = P\left(\lambda x, x, \frac{t^2}{|\lambda|}\right) =$
$P\left(x, x, \frac{t^2}{|\lambda|^2}\right) == N\left(x, \frac{t}{|\lambda|}\right), (\forall)\, t \geq 0, (\forall)\, \lambda \in \mathbb{K}^*$;

(N4) $N(x+t, t+s) \geq \min\{N(x,t), N(y,s)\}, (\forall)\, x, y \in H, (\forall)\, t, s \geq 0$.
If $t = 0$ or $s = 0$ the previous inequality is obvious. We asume that $t, s > 0$.

$$\begin{aligned}
N(x+y, t+s) &= P\left(x+y, x+y, (t+s)^2\right) = \\
&= P(x+y, x+y, t^2+s^2+ts+ts) \geq \\
&\geq P\left(x, x+y, t^2+ts\right) \wedge P\left(y, x+y, s^2+ts\right) \geq \\
&\geq P\left(x, x, t^2\right) \wedge P(x, y, ts) \wedge P(y, x, ts) \wedge P\left(y, y, s^2\right) = \\
&= P\left(x, x, t^2\right) \wedge P\left(y, y, s^2\right) = \\
&= \min\{N(x,t), N(y,s)\};
\end{aligned}$$

(N5) From **(FIP6)** it result that $N(x,\cdot)$ is left continuous and $\lim_{t \to \infty} N(x,t) = 1$.

4. Conclusion

By examining the available literature on fuzzy inner product space approaches, we presented an innovative solution to the issue.So, we have laid the groundwork for future study of the problems posed by fuzzy Hilbert space (FHS) theory, including the investigation of possible connections between the Pythagorean theorem, the parallelogram rule, and other orthogonality concerns in a fuzzy setting.Consequently, it is required to investigate bounded and linear functions in a fuzzy Hilbert space. In this approach, the study of continuous and limited functions in a fuzzy Banach space has advanced significantly.

First, the next sections will demonstrate an interpellation of a frequent and finite feature on an FHS. With this notion, we may delve into important operator classifications, including self-joint operations, regular carriers, and unitary operators. As we return to spectral theory, we will work on creating a functional analytic calculus.Finally, we intend to investigate the orthogonality in the context of the fuzzy Hilbert space. Without resorting to indirect approaches like a breakdown of the fuzzy Hilbert space, we can directly investigate the features of the self-adjoint projections.

References

1) Katsaras. Fuzzy topological vector spaces I. *Fuzzy Sets Syst.* **1982**, *7*, 84–96.
2) Katsaras. Fuzzy topological vector spaces II. *Fuzzy Sets Syst.* **1985**, *13*, 144–155.
3) Felbin. Finite dimensional fuzzy normed linear space. *Fuzzy Sets Syst.* **1993**, *49*, 240–249.
4) Bag. Finite dimensional fuzzy normed linear spaces. *J. Fuzzy Math.* **2004**, *12*, 688–706.
5) Vaezpour. Fuzzy Banach spaces. *J. Appl. Math. Comput.* **2006**, *18*, 476–485.