IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES

ISSN PRINT 2319 1775 Online 2320 7876

Research paper **Volume 11, Sp.Iss 7 , 2022**

A NUMERICAL SOLUTION TO SPDDE WITH DUAL LAYER USING FITTED METHOD

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DOI : 10.48047/IJFANS/11/S7/009

Abstract

This study focuses on the numerical solution of a challenging differential-difference equation within a dual boundary layer domain, employing an innovative fitted method. The equation is initially transformed into an ordinary singularly perturbed problem through a Taylor series expansion procedure. Subsequently, a three-term scheme is developed using finite differences, and the resulting tridiagonal system of equations is efficiently solved using the Thomas Algorithm. The study meticulously analyzes the accuracy of the solution by tabulating maximum absolute errors and employs graphical representations to illustrate the influence of fitting parameters on the layer structure. The research not only provides a practical numerical solution for complex problems but also contributes valuable insights into parameter sensitivity, facilitating precise adjustments for real-world applications. Additionally, the study advances the understanding of mathematical techniques, showcasing their adaptability in solving intricate problems encountered across various disciplines.

Key words: Singular Perturbation, Differential-Difference Equations, Dual Layer, Positive shift, Negative Shift.

1. Introduction

In the realm of control systems, the ubiquitous presence of time delays cannot be overlooked, stemming from the finite duration required for information sensing and subsequent response. This inherent characteristic leads to the formulation of singularly perturbed differential-difference equations (SPDDEs). These equations, where ordinary differential equations feature a small positive parameter multiplying the highest derivative and include at least one shift term (delay or advance), constitute a pivotal area of study in scientific and engineering domains.

The intrigue of SPDDEs lies in their multi-scale nature; they exhibit thin transition layers where solutions undergo rapid variation, while maintaining stability away from these layers, where variations occur at a slower pace. This intricate behavior renders SPDDEs fundamental to

ISSN PRINT 2319 1775 Online 2320 7876
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theoretical explorations and practical applications across various fields, including control theory(M.W. Derstine*et al.*,1982), physiology(K. Ikeda *et al.,1982)* and neural networks(M.K. Kadalbajoo, K.K. Sharma, 2005), among others.

Previous research efforts have delved into the complexities of SPDDEs, utilizing diverse techniques to unravel their intricate structure. Literature has explored approximate solutions employing methods such as matched asymptotic expansions and Laplace transforms, facilitating an in-depth understanding of the layer structures inherent in these differential-difference equations. Researchers have also proposed innovative numerical integration techniques, ranging from Numerov's difference scheme to exponential fitted methods, offering robust solutions tailored to specific types of SPDDEs. Furthermore, advancements in parametric spline schemes and fitted finite difference methods have expanded the toolkit for addressing nonlinear SPDDEs, providing a comprehensive approach to deciphering and solving these complex problems.

This introduction sets the stage for a comprehensive exploration of SPDDEs, highlighting their significance, prevalence in real-world scenarios, and the diverse methodologies employed to comprehend and solve them. Through this study, we delve into the intricate world of SPDDEs, aiming to contribute to the existing body of knowledge and enhance our understanding of these intriguing equations.

2. Description of the method

Consider singularly perturbed differential-difference equation of the form:

$$\varepsilon^2 u''(t) + b(t)u(t-\delta) + c(t)u(t) + d(t)u(t+\eta) = f(t) \qquad (1)$$

$\forall t \in (0,1)$ subject to the interval and boundary conditions

$$u(t) = \varphi(t) \text{on } -\delta \le t \le 0 \qquad (2)$$

$$u(t) = \gamma(t) \text{on } 1 \le t \le 1+\eta \qquad (3)$$

where

$a(t), b(t), c(t), d(t), f(t), \varphi(t)$ and $\gamma(t)$are sufficiently smooth functions on (0, 1), 0< ε << 1 is the perturbation parameter and $0 < \delta = O(\varepsilon)$ and $0 < \eta = O(\varepsilon)$ are the delay (negative shift) and the advance(positive shift) parameters respectively.

By using Taylor's expansion in the neighbourhood of the point *t*, we have

$$u(t-\delta) \approx u(t) - \delta u'(t) \qquad (4)$$

$$u(t+\eta) \approx u(t) + \eta u'(t) \qquad (5)$$

Using Eq. (4) and Eq. (5) in Eq. (1) we get an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\varepsilon^2 u''(t) + p(t)u'(t) + q(t)u(t) = f(t) \qquad (6)$$

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$$u(0) = \varphi(0) = \varphi_o \quad (7)$$

$$u(1) = \gamma(1) = \gamma_1 \quad (8)$$

where

$$p(t) = d(t)\eta - b(t)\delta \quad (9)$$

$$q(t) = b(t) + c(t) + d(t) \quad (10)$$

Since $0 < \delta << 1$and $0 < \eta << 1$, The transition from Eq.(1) to Eq.(6) is admissible. Further details on the validity of this transition is found in El'sgol'ts and Norkin. If $q(t) \leq 0$ on the interval [0,1], after that the solution of Eq.(1) exhibits boundary layers at each edges of the interval [0,1], whereas it exhibits oscillatory behaviour for $q(t) > 0$. Now, we considered duallayer problems.

From the theory of singular perturbations, the solution of Eqs. (6) -(8) is of the form

$$u(t) = u_o(x) + \frac{p(o)}{p(t)}(\varphi_o - u_o(o))e^{-\int_o^t\left(\frac{p(t)}{\varepsilon^2} - \frac{q(t)}{p(t)}\right)dx} + O(\varepsilon) \quad (11)$$

Where $u_o(t)$ is the solution of

$$p(t)u_o'(t) + q(t)u_o(t) = f(t),\ u_o(1) = \gamma_1 \quad (12)$$

Using the Taylor's series expansion for *p(t)* and *q(t)* about the point '*t* = 0' and limiting to their first terms, Eq. (11) becomes,

$$u(t) = u_o(t) + (\varphi_o - u_o(0))e^{-\left(\frac{p(o)}{\varepsilon^2} - \frac{q(o)}{p(o)}\right)t} + O(\varepsilon) \quad (13)$$

on discretizing the interval [0,1] into N equal subintervals of step size $\hbar = \frac{1}{N}$ to make sure that $t_i = i\hbar, i$ =0, 1, 2,.... ,*N*.

From Eq.(13), we have

$$u(ih) = u_o(ih) + (\varphi_o - u_o(0))e^{-\left(\frac{p(o)}{\varepsilon^2} - \frac{q(o)}{p(o)}\right)ih} + O(\varepsilon)$$

Therefore

$$\lim_{\hbar \to o} u(i\hbar) = u_o(0) + (\varphi_o - u_o(o))e^{-\left(\frac{p^2(o) - \varepsilon q(o)}{p(o)}\right)i\rho} \quad (14)$$

where $\rho = \frac{\hbar}{\varepsilon^2}$

Assuming that $u(t)$ is continuously differentiable in the interval [0,1] and applying Taylor's series expansion for $u(t_{i+1})$ and $u(t_{i-1})$, we have:

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$$u(t_{i+1}) = u_{i+1} = u_i + h u_i^{'} + \frac{h^2}{2!}u_i^{''} + \frac{h^3}{3!}u_i^{'''} + \frac{h^4}{4!}u_i^{(4)} + \frac{h^5}{5!}u_i^{(5)} + \frac{h^6}{6!}u_i^{(6)} + \frac{h^7}{7!}u_i^{(7)} + \frac{h^8}{8!}u_i^{(8)} + O(h^9)$$

$$u(t_{i-1}) = u_{i-1} = u_i - h u_i^{'} + \frac{h^2}{2!}u_i^{''} - \frac{h^3}{3!}u_i^{'''} + \frac{h^4}{4!}u_i^{(4)} - \frac{h^5}{5!}u_i^{(5)} + \frac{h^6}{6!}u_i^{(6)} - \frac{h^7}{7!}u_i^{(7)} + \frac{h^8}{8!}u_i^{(8)} - O(h^9)$$

From the finite differences, we have

$$u_{i-1} - 2u_i + u_{i+1} = \frac{2h^2}{2!}u_i^{''} + \frac{2h^4}{4!}u_i^{(4)} + \frac{2h^6}{6!}u_i^{(6)} + \frac{2h^8}{8!}u_i^{(8)} + O(h^{10}) \qquad (15)$$

Now we have the relation:

$$u_{i-1}^{''} - 2u_i^{''} + u_{i+1}^{''} = \frac{2h^2}{2!}u_i^{(4)} + \frac{2h^4}{4!}u_i^{(6)} + \frac{2h^6}{6!}u_i^{(8)} + \frac{2h^8}{8!}u_i^{(10)} + O(h^{12})$$

Substituting $\frac{h^4}{12}u_i^{(6)}$ from the above equation in Eq.(15), we have

$$u_{i-1} - 2u_i + u_{i+1} = h^2u_i^{''} + \frac{h^4}{12}u_i^{(4)} + \frac{h^2}{30}\left[u_{i-1}^{''} - 2u_i^{''} + u_{i+1}^{''} - h^2u_i^{(4)} - \frac{h^6}{360}u_i^{(8)}\right] + \frac{2h^8}{8!} + O(h^{10})$$

$$u_{i-1} - 2u_i + u_{i+1} = h^2\left[u_i^{''} + \frac{1}{30}(u_{i-1}^{''} - 2u_i^{''} + u_{i+1}^{''})\right] + \frac{h^4}{12}u_i^{(4)} - \frac{h^4}{30}u_i^{(4)} - \frac{h^6}{10800}u_i^{(8)} + \frac{2h^8}{8!}u_i^{(8)} + O(h^{10})$$

and

$$u_{i-1} - 2u_i + u_{i+1} = \frac{h^2}{30}(u_{i-1}^{''} + 28u_i^{''} + u_{i+1}^{''}) + R, \qquad (16)$$

Where $R = \frac{h^4}{20}u_i^{(4)} + \frac{13h^6}{302400}u_i^{(8)} + O(h^{10})$

Now from the Eq. (6), we have

$$\varepsilon u_{i+1}^{''} = -p_{i+1}u_{i+1}^{'} - q_{i+1}u_{i+1} + f_{i+1}$$

$$\varepsilon u_i^{''} = -p_iu_i^{'} - q_iu_i + f_i$$

$$\varepsilon u_{i-1}^{''} = -p_{i-1}u_{i-1}^{'} - q_{i-1}u_{i-1} + f_{i-1} \qquad (17)$$

Using the following three-point approximations for first-order derivatives:

$$u_{i+1}^{'} \simeq \frac{u_{i-1} - 4u_i + 3u_{i+1}}{2h}$$

$$u'_i \simeq \frac{u_{i+1} - u_{i-1}}{2h}$$

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$$u'_{i-1} \simeq \frac{-3u_{i-1}+4u_i-u_{i+1}}{2h} \tag{18}$$

Substituting Eq. (17) and Eq.(18) in Eq.(16) and simplifying we get

$$\varepsilon\left(\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}\right)+\frac{p_{i-1}}{60h}[-3u_{i-1}+4u_i-u_{i+1}]+\frac{28p_i}{60h}[u_{i+1}-u_{i-1}]+\frac{p_{i+1}}{60h}[u_{i-1}-4u_i+3u_{i+1}]+\frac{q_{i-1}}{30}y_{i-1}+\frac{28q_i}{30}y_i+\frac{q_{i+1}}{30}y_{i+1}=\frac{[f_{i-1}+28f_i+f_{i+1}]}{30} \tag{19}$$

The tridiagonal system Eq. (19) is given by

$$A_i u_{i-1} - B_i u_i + C_i u_{i+1} = D_i,$$

for $i = 1,2,\ldots,N-1$ (20)

where

$$A_i = \frac{\varepsilon}{h^2} - \frac{3p_{i-1}}{60h} + \frac{q_{i-1}}{30} - \frac{28p_i}{60h} + \frac{p_{i+1}}{60h}$$

$$B_i = \frac{2\varepsilon}{h^2} - \frac{4p_{i-1}}{60h} - \frac{28q_i}{30} + \frac{4p_{i+1}}{60h}$$

$$C_i = \frac{\varepsilon}{h^2} - \frac{p_{i-1}}{60h} + \frac{q_{i+1}}{30} + \frac{28p_i}{60h} + \frac{3p_{i+1}}{60h}$$

$$D_i = \frac{1}{30}[f_{i-1} + 28f_i + f_{i+1}]$$

We use the Thomas algorithm to solve this tridiagonal system Eq. (20).

3. Numerical examples

The proposed method is validated on examples of the similar type of Eqs. (1)- (3). For the singularly perturbed differential-difference equation:

$$\varepsilon^2 u''(x) + b(x)u(x-\delta) + c(x)u(x) + d(x)u(x+\eta) = f(x)$$

$\forall x \in (0,1)$ and subject to the interval and boundary conditions

$$u(x) = \varphi(x), \quad on -\delta \le x \le 0$$

$$u(x) = \gamma(x), \quad on \quad 1 \le x \le 1+\eta$$

with constant coefficients(i.e,

$$b(x) = b, c(x) = c, d(x) = d, f(x) = f, \varphi(x) = \varphi \; and \; \gamma(x) = \gamma)$$

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is given by $u(x) = \frac{[(1-b-c-d)e^{m_2}-1]e^{m_1 x}-[(1-b-c-d)e^{m_1}-1]e^{m_2 x}}{(b+c+d)(e^{m_1}-e^{m_2})} + \frac{1}{b+c+d}$

where

$$m_1 = \frac{(b\delta - d\eta)+\sqrt{(d\eta - b\delta)^2 - 4\varepsilon^2(b+c+d)}}{2\varepsilon^2}, \qquad m_2 = \frac{(b\delta - d\eta)-\sqrt{(d\eta - b\delta)^2 - 4\varepsilon^2(b+c+d)}}{2\varepsilon^2}$$

Example 1. Consider the SPPDE with constant coefficients
$\varepsilon^2 u''(x) - 2u(x-\delta) - u(x) - 2u(x+\eta) = 1\,, \quad \varphi(x) = 1, \qquad \gamma(x) = 0$
The results are shown in Table 1 and 2 & Figure 1 and 2.
Example 2. Consider the SPPDE with constant coefficients
$\varepsilon^2 u''(x) + 0.25u(x-\delta) - u(x) + 0.25u(x+\eta) = 1\,, \quad \varphi(x) = 1, \qquad \gamma(x) = 0$
The results are shown in Table 3 and 4 & Figure 3 and 4.

Table 1. In solution of Example 1, the numerical results for N=100, $\varepsilon = 0.1$ and $\delta = 0.07$

x	$\eta = 0$		$\eta = 0.03$		$\eta = 0.06$	
	Num. Solution	Exact Solution	Num. Solution	Exact Solution	Num. Solution	Exact Solution
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.02	0.45242458	0.45264934	0.50300731	0.50344253	0.55169184	0.55214441
0.04	0.15471487	0.15495931	0.21184940	0.21235949	0.27086719	0.27143434
0.06	-0.00714606	-0.00694668	0.04127762	0.04172600	0.09495586	0.09548892
0.08	-0.09514776	-0.09500320	-0.05865005	-0.05829970	-0.01523674	-0.01479139
0.10	-0.14299314	-0.14289488	-0.11719162	-0.11693498	-0.08426247	-0.08391365
0.20	-0.19729146	-0.19728211	-0.19428558	-0.19425011	-0.18883735	-0.18876996
0.40	-0.19998330	-0.19998323	-0.19997011	-0.19996975	-0.19989563	-0.19989436
0.60	-0.19971798	-0.19971742	-0.19988711	-0.19988642	-0.19996039	-0.19995998
0.80	-0.19248997	-0.19248250	-0.19525107	-0.19523670	-0.19721989	-0.19720588
0.90	-0.16124429	-0.16122502	-0.16918139	-0.16913482	-0.17641992	-0.17636058
0.92	-0.14618856	-0.14616716	-0.15520223	-0.15514807	-0.16383885	-0.16376607
0.94	-0.12528403	-0.12526175	-0.13488218	-0.13482316	-0.14454520	-0.14446151
0.96	-0.09625855	-0.09623793	-0.10534508	-0.10528788	-0.11495748	-0.11487195
0.98	-0.05595734	-0.05594302	-0.06241008	-0.06236852	-0.06958335	-0.06951778
1.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
Maximum Error : 3.0469e-04			1.8305e-04		4.8258e-04	

Table 2. In solution of Example 1, the maximum absolute errors for $\delta = \varepsilon^2$ and $\eta = 2\varepsilon^2$

$\varepsilon \backslash N$	128	256	512	1024	2048
0.1	3.6651e-04	9.1836e-05	2.2972e-05	5.7438e-06	1.4360e-06
0.01	2.1734e-02	7.7833e-03	2.0641e-03	5.2711e-04	1.3272e-04

Table 3. In solution of Example 2, the numerical results for N=100, $\varepsilon = 0.01$ and $\eta = 0.007$

x	$\delta = 0$		$\delta = 0.003$		$\delta = 0.006$	
	Num. Solution	Exact Solution	Num. Solution	Exact Solution	Num. Solution	Exact Solution
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.02	-1.38767494	-1.39431045	-1.33302029	-1.34238288	-1.27719916	-1.28881467
0.04	-1.87501934	-1.87771339	-1.85171269	-1.85584657	-1.82585298	-1.83140514
0.06	-1.97449040	-1.97531076	-1.96703179	-1.96840075	-1.95804213	-1.96003260
0.08	-1.99479328	-1.99501533	-1.99267029	-1.99307326	-1.98989094	-1.99052526
0.10	-1.99893726	-1.99899361	-1.99837041	-1.99848162	-1.99756439	-1.99775390

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0.20	-1.99999962	-1.99999966	-1.99999911	-1.99999923	-1.99999802	-1.99999832
0.40	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000
0.60	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000
0.80	-1.99999110	-1.99999255	-1.99999548	-1.99999621	-1.99999781	-1.99999815
0.90	-1.99578031	-1.99613909	-1.99699376	-1.99724841	-1.99790619	-1.99807730
0.92	-1.98553127	-1.98652411	-1.98896875	-1.98972284	-1.99174047	-1.99228502
0.94	-1.95038873	-1.95296451	-1.95952135	-1.96161494	-1.96741828	-1.96904301
0.96	-1.82988986	-1.83583000	-1.85146549	-1.85663224	-1.87147350	-1.87578258
0.98	-1.41671595	-1.42699041	-1.45495961	-1.46452309	-1.49299605	-1.50156761
1.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
Maximum Error : 1.0274e-02			9.9633e-03		1.1880e-02	

Table 4. In solution of Example 2, the maximum absolute errors for $\delta = \varepsilon^2$ and $\eta = 2\varepsilon^2$

$\varepsilon \backslash N$	128	256	512	1024	2048
0.1	8.2492e-05	2.0630e-05	5.1581e-06	1.2896e-06	3.2239e-07
0.01	8.0762e-03	2.0682e-03	5.2272e-04	1.3088e-04	3.2750e-05

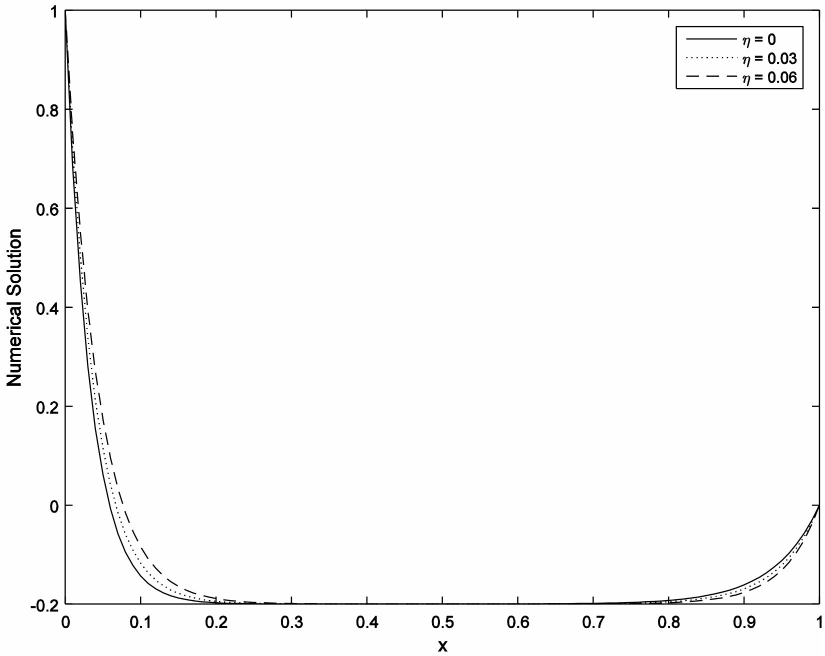

Fig. 1. Numerical solution of Example 1 for N=100, $\varepsilon = 0.1$ *and* $\delta = 0.07$

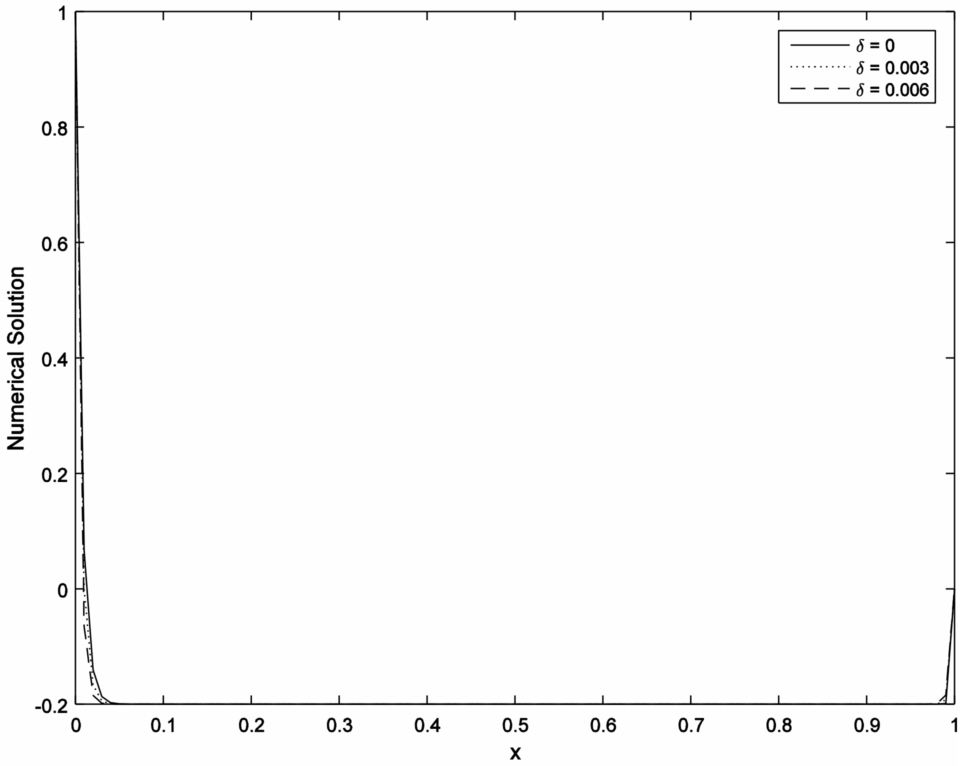

Fig. 2. Numerical solution of Example 1 for N=100, $\varepsilon = 0.01$ *and* $\eta = 0.007$

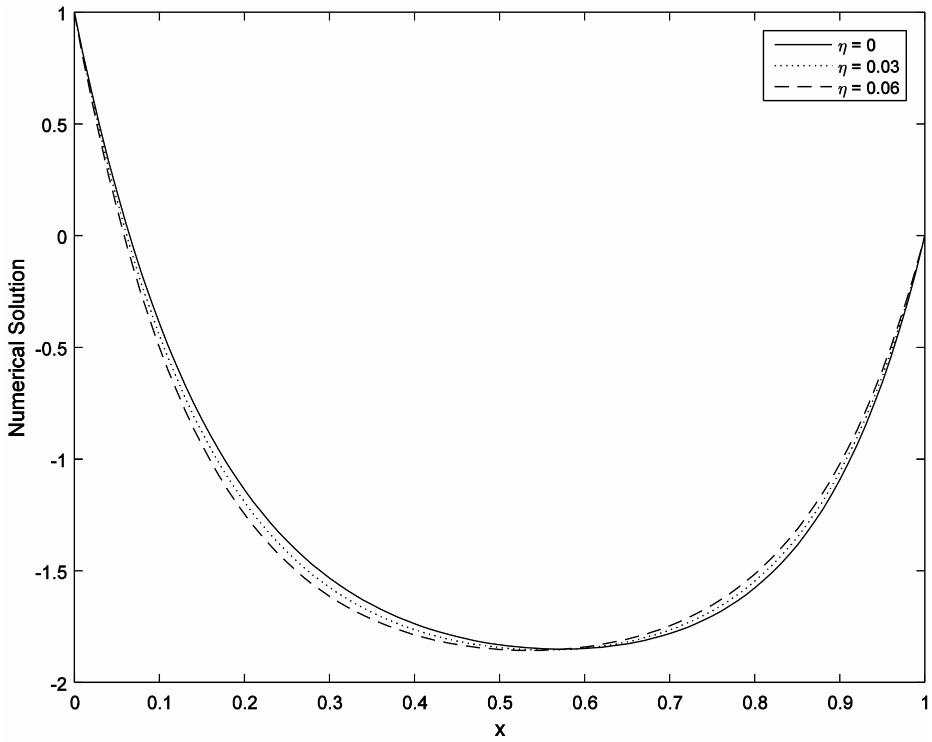

Fig. 3. Numerical solution of Example 2 for N=100, $\varepsilon = 0.1,$ *and* $\delta = 0.07$

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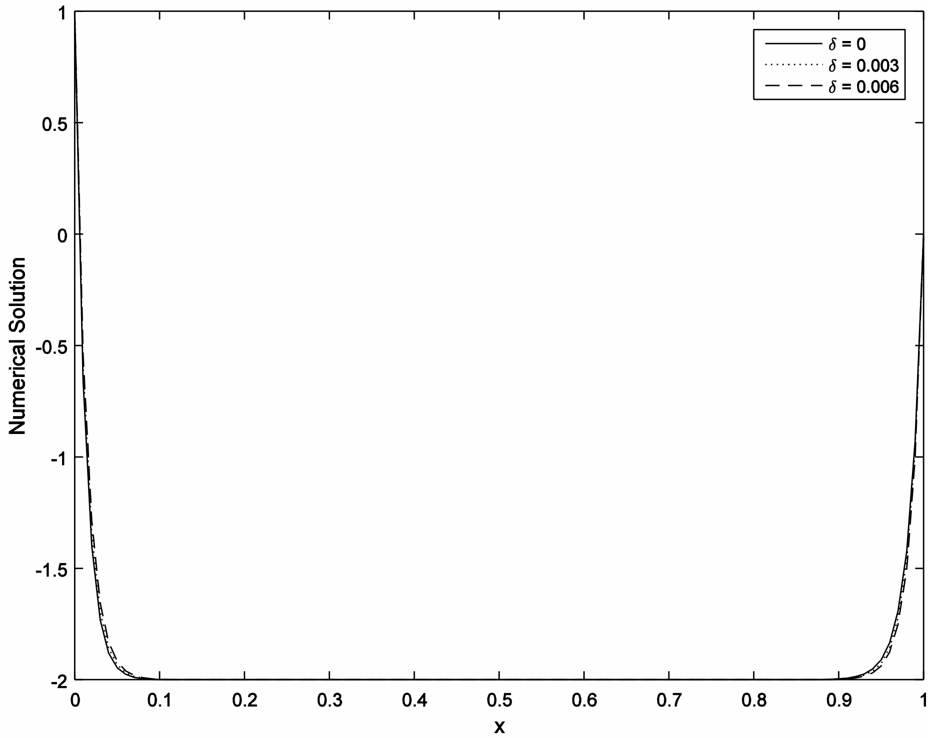

Fig. 4. Numerical Solution of Example 2 for N=100, $\varepsilon = 0.01, \text{and } \eta = 0.007$

4. Discussions and conclusion

This study has introduced and applied a tailored fitted method for solving singularly perturbed differential-difference equations displaying dual-layer behavior. Through a series of model problems involving variations in parameters such as ε, δ, η, and h, we systematically evaluated the performance of our method. By presenting the maximum absolute errors and computational orders for well-established examples from the literature, we conducted a comprehensive analysis. The comparison between our numerical solutions and exact solutions validated the accuracy and reliability of our proposed approach. The results unequivocally show that our method excels in approximating exact solutions, affirming its robustness and effectiveness in dealing with the intricacies of singularly perturbed differential-difference equations featuring dual-layer phenomena. This study underscores the practical applicability and potential of the fitted method in accurately capturing the behavior of complex systems governed by such equations.

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