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COMMON FIXED-POINT RESULTS FOR RATIONL TYPE CONTRACTIONS

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ABSTRACT
In this study, we demonstrate a few fixed-point results involving contractions of the rational type in the generalised 2-Banach space. Our findings broaden standard common fixed-point theorems of contractive type mappings in the new context.
Keywords
2-Banach Space, Generalized Banach Space, generalized 2-Banach Space, Common Fixed-Point, Altering Distance Function.

1. INTRODUCTION.
Banach Fixed-point theorem is also known as contraction mapping theorem or contraction mapping principle. Banach fixed-point theorem was named after Stefan Banach. Banach fixed-point theorem guarantees the existence and uniqueness of fixed points of certain self-maps. 2-norm and n-norm on a linear space was introduced by S. Gahler in 1963. The metric fixed-point theory is a vast field of study and it is capable of solving many mathematical equations. It has a wide range of applications in many fields of science. Czerwik needs an extension of metric space to overcome the problem of measurable functions with respect to a measure and their convergence. He proved a generalized fixed-point theorem in b-metric space [9,10]. The notion of b-metric spaces was introduced by Bakhtin [2] In 1989, which was formally defined by Czerwik [8] in 1993 with a view of generalizing Banach contraction principle. There are many authors who have worked on the generalization of fixed-point theorems in b-metric spaces. In particular, the extension of fixed-point theorem in generalized Banach space was studied by many researchers.

2. PRELIMIARIES.
Definition 2.1. [4] If $S \neq \varphi$ is a linear space having $a(\geq 0) \in \mathbb{R}$, let $\|.\|$ denotes a function from linear space S into $\mathbb{R}$ that satisfies the following axioms:

- For all $p \in S$, $\|p\| = 0$ if and only if $p = 0$
- For all $p, q \in S$, $\|p + q\| \leq a\{\|p\| + \|q\|\}$
- For all $p \in S, \alpha \in \mathbb{R}$, $\|\alpha p\| = |\alpha|\|p\|$

$\|p\|$ is called norm of p and $(S, \|.\|)$ is called generalized normed linear space. If for$a = 1$, it reduces to standard normed linear space.

Definition 2.2. [4] A Banach space $(S, \|.\|)$ is a normed vector space such that S is complete under the metric induced by the $\|.\|$.

Definition 2.3. [4] A linear generalized normed space in which every sequence is convergent is called generalized Banach Space.

Definition 2.4. [4] Let $(S, \|.\|)$ be a generalized normed linear space then the sequence $\{p_u\}$ in S is called Cauchy sequence if and only if for all $\epsilon > 0$, there exists $u(\epsilon) \in \mathbb{N}$ such that for each $t, u \geq u(\epsilon)$ we have $\|p_u - p_t\| < \epsilon$.

Definition 2.5. [4] Let $(S, \|.\|)$ be a generalized normed linear space then the sequence $\{p_u\}$ in S is called Convergent sequence if and only if there exists $p \in S$ such that for all $\epsilon > 0$, there exists $u(\epsilon) \in \mathbb{N}$ such that for each $u \geq u(\epsilon)$ we have $\|p_u - p\| < \epsilon$.

Definition 2.6. [4] The generalized Banach Space is Complete if every Cauchy sequence converges.

Lemma 2.1. [16] Let $(S, \|.\|)$ be a generalized Banach space and let $\{p_u\}$ be a Cauchy sequence in S such that $p_t \neq p_u$ whenever $t \neq u$. Then $\{p_u\}$ converges to atmost one point.

Proof:

Suppose that $\lim_{u\to\infty} p_u = p$ and $\lim_{u\to\infty} p_u = q$

Let us conversely assume that $p \neq q$.

Which implies p and q are distinct elements

Since, p_t and p_u are distinct elements, it is clear that there exists $\wp \in \mathbb{N}$ such that p_u is different from p and q for all $u > \wp$

For $t, u > \wp$, implies that

$$\|p - q\| \leq a\{\|p - p_u\| + \|p_u - q\|\}$$

Letting $u \to \infty$,

We have $\|p - q\| = 0$,

$\Rightarrow p = q$, which is a contradiction.

3. MAIN RESULTS

Theorem 3.1. Let $(S, \|.,.\|)$ be a generalized 2-Banach space with $a \geq 1$ and let $K, L: S \to S$ be an increasing mapping with respect to $' \leq '$ such that there exists an element $p_0 \in S$ with $p_0 \leq Kp_0$ and $p_0 \leq Lp_0$. Suppose that

$$a^2\|Kp - Lq, s\| \leq \lambda(\zeta[(p, q), s])$$

Where $\zeta[(p,q),s] = max\left\{\|p - q, s\|, \frac{\|p-Kp,s\|.\|q-Lq,s\|}{1+\|Kp-Lq,s\|}\right\}$ for $\lambda \in \Lambda$ and for all $p, q \in S$ with p, q comparable. Then K and L has a unique fixed point, if K and L is continuous. In addition, the set of fixed points of K and L is well ordered if and only if K and L has a unique common fixed point.

Proof:

Let $p_0 \in S, p_0 \leq Kp_0$ and $p_0 \leq Lp_0$

Also, K and L are increasing mapping. By induction, we obtain that

$$p_0 \leq Kp_0 \leq K^2p_0 \leq \cdots \leq K^u p_0 \leq K^{u+1}p_0 \leq \cdots$$
$$p_0 \leq Lp_0 \leq L^2p_0 \leq \cdots \leq L^u p_0 \leq L^{u+1}p_0 \leq \cdots$$

Define the sequence p_u by $p_{2u+1} = Kp_{2u}$ and $p_{2u+2} = Lp_{2u+1}$ for all $u \geq 0$.

Let $p_u = K^u p_0$ and $p_u = L^u p_0$, we have

$$p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_u \leq p_{u+1} \leq \cdots$$
$$a^2\|p_{2u+1} - p_{2u+2}, s\| = a^2\|Kp_{2u} - Lp_{2u+1}, s\|$$
$$\leq \lambda(\zeta[(p_{2u}, p_{2u+1}), s])$$

By Mathematical induction, we obtain

Uniqueness of common fixed point:

Now if w be another fixed point of K. Then $Kw = w$ then,

$$\|p^* - Kw, s\| \leq a[\|p^* - p_u, s\| + \|p_u - Kw, s\|]$$

Letting $u \to \infty$ and using continuity of K, we get

$$\lim_{u\to\infty} \|p^* - Kw, s\| \leq 0$$

Theorem 3.2. Let $(S, \|.,.\|)$ be a generalized 2-Banach space with $a \geq 1$ and let $K, L: S \to S$ be an increasing mapping with respect to $' \leq '$ such that there exists an element $p_0 \in S$ with $p_0 \leq Kp_0$ and $p_0 \leq Lp_0$. Also, K and L satisfies the following condition

$$\|Kp - Lq, s\| \leq \delta[\|p - q, s\|]\beta[(p, q), s] \quad (1)$$

For all $p, q \in S$ are comparable, where a function $\delta: [0, \infty) \to \left[0, \frac{1}{a}\right]$ satisfies the condition $\lim_{u\to\infty} \sup \delta(k_u) = \frac{1}{a}$ implies $\lim_{u\to\infty} k_u = 0$ and

$$\beta[(p,q),s] = \max\left\{\begin{array}{c} \|p-q,s\|, \\ \frac{\|p-Kp,s\|.\|q-Lq,s\|}{1+\|Kp-Lq,s\|}, \\ \frac{\|p-Kp,s\|.\|q-Lq,s\|}{1+\|p-q,s\|}, \\ \frac{\|p-Kp,s\|.\|p-Lq,s\|}{1+\|p-Kq,s\|.\|q-Lq,s\|} \end{array}\right\}$$

If K and L is continuous, then K and L has unique fixed-point.

Proof.

Let $p_0 \in S, p_0 \leq Kp_0$ and $p_0 \leq Lp_0$

Also, K and L are increasing mapping. By induction, we obtain that$p_0 \leq Kp_0 \leq K^2p_0 \leq \cdots \leq K^u p_0 \leq K^{u+1}p_0 \leq p_0 \leq Lp_0 \leq L^2p_0 \leq \cdots \leq L^u p_0 \leq L^{u+1}p_0 \leq \cdots$

We will prove that $\lim_{u\to\infty} \|p_{2u+1} - p_{2u+2}, s\| = 0$

since, $p_{2u+1} \leq p_{2u+1}$ $\forall\, u \in \mathbb{N}$. By (6), we have$\|p_{2u+1} - p_{2u+2}, s\| = \|Kp_{2u} - Lp_{2u+1}, s\|$

(Since $a > 1$) hence $v = 0$

$$\Longrightarrow \lim_{u\to\infty} \|p_{2u+1} - p_{2u+2}, s\|$$

First suppose that $p_{2u} = p_{2t}$ for some $u > t$, so we have, $p_{2u+1} = Kp_{2u} = Kp_{2t} = p_{2t+1}, p_{2u+1} = Lp_{2u} = Lp_{2t} = p_{2t+1}$

By continuing this process,

$$p_{2u+\wp} = p_{2t+\wp} \text{ for } \wp \in \mathbb{N}$$

Thus, we can assume that $p_{2u} \neq p_{2t}$ for $u \neq t$.

We deduce that $\lim_{u,t\to\infty} \|p_{2u} - p_{2t}, s\| = 0$

Consequently, $\{p_{2u}\}$ is a Cauchy sequence in K and so is $\{p_u\}$

$\therefore$ p^* is the unique common fixed point of K and L

This completes the proof.

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