THE UPPER EDGE TO EDGE DETOUR DOMINATION NUMBER OF A GRAPH

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ABSTRACT—An edge-to-edge detour dominating set S in a connected graph G is called a minimal edge-to-edge detour dominating set if no proper subset of S is an edge-to-edge detour dominating set of G. The upper edge-to-edge detour domination number $\gamma_{d_{ee}}^{+}(G)$ of G is the maximum cardinality of a minimal edge-to-edge detour dominating set of G. Some general properties satisfied by this concept are studied. The upper edge-to-edge detour domination number of some standard graphs are determined. It is shown that for any two positive integers a and b with $2 \le a \le b$, there exists a connected graph G such that $\gamma_{d_{ee}}(G) = a$ and $\gamma_{d_{ee}}^{+}(G) = b$.

Keywords—detour number, domination number, edge-to-edge detour number, edge-to-edge detour domination number.

I INTRODUCTION

By a graph $G = (V, E)$, we mean a finite,undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [3]. Two vertices u and v are said to be adjacent if uv is an edge of G. Two edges of G are said to be adjacent if they have a common vertex. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u−v path of length $d(u, v)$ is called an u−v geodesic.

The detour distance $D(u, v)$ between two vertices u and v in a connected graph G from u to v is define as the length of a longest u-v path in G. An u-v path of length $D(u, v)$ is called an u-v detour. A vertex x is said to lie on an u-v detour P if x is a vertex of P including the vertices u and v. A detour set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a detour joining some pair of vertices in S. The closed detour $I_D[u, v]$ consists of all the vertices lying on some u-v detour of G including the vertices u and v. The detour number $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called minimum detour set of G or a dn-set of G. These concept were studied in [2,3,5,6]. A set $S \subseteq E$ is called an *edge-to-edge detour set* of G if every edge of G is an element of S or lies on a detour joining a pair of edges of S. The *edge-to-edge detour number* $d_{ee}(G)$ of G is the minimum cardinality of its edge-to- edge detour sets and any edge-to- edge detour set of cardinality $d_{ee}(G)$ is said to be a d_{ee}-set of G. The edge-to-edge detour number of a graph was studied in [7]. A set $D \subseteq V(G)$ is a dominating set of G if for every $v \in V(G) \setminus D$ is adjacent to some vertex in D. A dominating set D is said to be minimal if no proper subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. Any dominating set of cardinality $\gamma(G)$ is a γ-set of G. The domination number of a graph was studied in [4]. Let $G = (V, E)$ be a connected graph with at least three vertices.

Theorem 1.1 [5] Every end-edge of a connected graph G belongs to every edge-to-edge detour set of G. The Upper Edge-to-Edge Detour Domination Number of a Graph

Definition 2.1. An edge-to-edge detour dominating set S in a connected graph G is called a minimal edge-to-edge detour dominating set if no proper subset of S is an edge-to-edge detour dominating set of G. The upper edge-to-edge detour domination number $\gamma_{d_{ee}}^{+}(G)$ of G is the maximum cardinality of a minimal edge-to-edge detour dominating set of G.

Remark 2.3. Every minimum edge-to-edge detour dominating set of G is a minimal edge-to-edge detour dominating set of G and the converse is not true. For the graph G given in Figure 2.1, $S =$

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$\{v_1v_2, v_7v_8, v_3v_4, v_5v_6\}$ is a minimal edge-to-edge detour dominating set but not a minimum edge-to-edge detour dominating set of G.

Theorem 2.4. For a connected graph G, $2 \leq \gamma_{d_{ee}}(G) \leq \gamma_{d_{ee}}^+(G) \leq q$.

Proof. Any edge-to-edge detour dominating set needs at least two edges and so $\gamma_{d_{ee}}(G) \geq 2$. Since every minimal edge-to-edge detour dominating set of G is an edge-to-edge detour dominating set of G, $\gamma_{d_{ee}}(G) \leq \gamma_{d_{ee}}^+(G)$. Also since $E(G)$ is an edge-to-edge detour dominating set of G, it is clear that $\gamma_{d_{ee}}^+(G) \leq q$. Thus $2 \leq \gamma_{d_{ee}}(G) \leq \gamma_{d_{ee}}^+(G) \leq q$.

Remark 2.5. The bounds in Theorem 2.4 are sharp. For the graph $G = P_4$, $\gamma_{d_{ee}}(G) = 2$, $\gamma_{d_{ee}}^+(G) = 2$ and so $2 = \gamma_{d_{ee}}(G) = \gamma_{d_{ee}}^+(G)$ and also $\gamma_{d_{ee}}^+(K_{1,q}) = q$ for $q \geq 2$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 2.1, $q = 9$, $\gamma_{d_{ee}}(G) = 3$ and $\gamma_{d_{ee}}^+(G) = 4$ so that $2 < \gamma_{d_{ee}}(G) < \gamma_{d_{ee}}^+(G) < q$.

Theorem 2.6. For a connected graph G of size $q \geq 2$, $\gamma_{d_{ee}}(G) = q$ if and only if $\gamma_{d_{ee}}^+(G) = q$.

Proof. Let $\gamma_{d_{ee}}^+(G) = q$. Then $S = E(G)$ is the unique minimal edge-to-edge detour dominating set of G. Since no proper subset of S is an edge-to-edge detour dominating set of G, it is clear that S is the unique minimum edge-to-edge detour dominating set of G and so $\gamma_{d_{ee}}(G) = q$ The converse follows from Theorem 2.4.

Corollary 2.7. For a connected graph G of size $q \geq 2$, the following are equivalent.

i) $\gamma_{d_{ee}}(G) = q$

ii) $\gamma_{d_{ee}}^+(G) = q$

iii) $G = K_{1,p}$.

Proof. This follows from Theorem 2.6.

Theorem 2.8. For complete graph $G = K_p$, $\gamma_{d_{ee}}^+(G) = \frac{p}{2}$.

Proof. Let S be any set of $\frac{p}{2}$ independent edges of K_n. Then S is an edge dominating set of G. We have to prove S is a minimal edge-to-edge detour dominating set of G. If not, let $X \subset S$ be such that X is a minimal edge-to-edge detour dominating set of G. Therefore there exists at least one edge e of S with $e \notin X$. Hence e does not lies on a detour joining a pair of edges of X and so X is not an edge-to-edge detour dominating set of G, which is a contradiction. Hence S is a minimal edge-to-edge detour dominating set of G. Therefore $\gamma_{d_{ee}}^+(G) \geq \frac{p}{2}$. Next we prove that $\gamma_{d_{ee}}^+(G) = \frac{p}{2}$. If not, suppose $\gamma_{d_{ee}}^+(G) > \frac{p}{2}$. Then there exists a minimal edge-to-edge detour dominating set S' with $|S'| \geq \left\lceil \frac{p}{2} \right\rceil + 1$.

Case 1. Suppose that every edge of S' is incident with a vertex of G. Then S' is an edge dominating set of G and since $d(e, f) = 0 \; \forall \; e, f \in S'$, S' is not an edge-to-edge detour set of G and so S' is not an edge-to-edge detour dominating set of G, which is a contradiction.

Case 2. Suppose that some edges of S' are incident with a vertex of G and some of them are independent. Let Y be the independent edges of S' and Z be the independent edges of G. Hence $|Z| = \frac{p}{2}$ and $|Y| < \frac{p}{2}$. Then there exists at least one edge e such that $e \in Z$ and $e \notin Y$. Hence $e \notin S'$. Then e does not lie on a detour joining a pair of edges of S' and so S' is not an edge-to-edge detour dominating set of G, which is a contradiction. Hence $\gamma_{d_{ee}}^+(G) = \frac{p}{2}$.

Theorem 2.9 . For any two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $\gamma_{d_{ee}}(G) = a$ and $\gamma_{d_{ee}}^+(G) = b$.

Proof. For $a = b$, let $G = K_{1,a}$. Then by Theorem 2.6, $\gamma_{d_{ee}}(G) = \gamma_{d_{ee}}^+(G) = a$. Therefore let $2 \leq a < b$. Let $P: u, v, w, x$ be a path of order 4. Let G be a graph obtained from P by adding new vertices $y_1, y_2, \ldots, y_{b-a}$ and

$z_1, z_2, \dots, z_{a-1}$ and join x with each $z_i (1 \le i \le a-1)$ and each y_i $(1 \le i \le b-a)$ and join u with each y_i $(1 \le i \le b-a)$. The graph G is shown in Figure 2.2First we claim that $\gamma_{d_{ee}}(G) = a$. Let $Z = \{xz_1, xz_2, \dots, xz_{a-1}\}$. By Theorem 1.1, Z is a subset of every edge-to-edge detour dominating set of G, and so $\gamma_{d_{ee}}(G) \ge a$. Now $Z' = Z \cup \{uv\}$ is an edge-to-edge detour dominating set of G so that $\gamma_{d_{ee}}(G) = a$. Next we show that $\gamma^+_{d_{ee}}(G) = b$. Let $S = \{uy_1, uy_2, \dots, uy_{b-a}\}$. Then $D = Z \cup S \cup \{vw\}$ is an edge-to-edge detour dominating set of G. We will show that D is a minimal edge-to-edge detour dominating set of G. Let D' be any proper subset of D. Then there exists at least one edge $e \in D$ such that $e \notin D'$. By Theorem 1.1 $e \ne xz_i\{1 \le i \le a-1\}$. Suppose that $e = vw$, then e is not dominated by any edge of G. Also it is easily seen that the edges uv, vw and wx does not lies on the detour joining a pair of edges of G. We claim that $\gamma^+_{d_{ee}}(G) = b$. Suppose that there exists a minimal edge-to-edge detour dominating set T of G such that $|T| > b$. By Theorem 5.5, Z is a subset of T. Since T is a minimal edge-to-edge detour dominating set, Z' is not a subset of T and D is not a subset of T and $uv \notin T$. Let $S' = \{xy_1, xy_2, \dots, xy_{b-a}\}$. Then T contains some edges of S and some edges of S'. Suppose xy_i $(1 \le i \le b-a)$ does not belongs to T. Then the edges $xy_1, xy_2, \dots, xy_{b-a}$ are dominated by some edges of T but not lies on detour joining a pair of edges of G. Suppose $uy_i(1 \le i \le b-a)$ does not belongs to T. Then the edges $uy_1, uy_2, \dots, uy_{b-a}$ are not dominated by any edges of T and not lies on detour joining a pair of edges of G. Therefore T is not an edge-to-edge detour dominating set. Hence $\gamma^+_{d_{ee}}(G) = b$.

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