IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES

ISSN PRINT 2319 1775 Online 2320 7876

Research Paper UGC CARE Listed (Group-I) Journal Volume 11, Iss 10, Oct 2022

Covering Polynomial of $K_{1,n} \times P_2$

Helen Sheeba M

Assistant Professor, *Department of Mathematics, Pioneer Kumarasamy College, Nagercoil.*

(Affiliated to Manonmaniam Sundaranar University, Tirunelveli)

Mail: hsheeba67@gmail.com

ABSTRACT:

The vertex cover polynomial of a graph G of order n has been already introduced in [3]. It is defined as the polynomial, $C(G, x) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)x^i$, where

C(G,i) is the number of vertex covering sets of G of size i and β (G) is the covering number of G. In this paper we have established a general formula for finding the vertex Cover Polynomial to the product of P_2 with the complete graph $K_{n,}$($ie\ G = K_{1,n} \times P_2$). The coefficient of the polynomial satisfies some identities. Also we have proved that the coefficient of the vertex cover, polynomial C(G,x)is log-Concave.

***Key words**: Vertex covering set, Vertex covering number, Vertex cover polynomial.*

Introduction: 1

Let G = (V, E) be a simple graph. For any vertex $v \in$ V, the open neighborhood of $v \in$ V is the set N(v) = {u ϵ V /uv $v \in$ E} and the closed neighbourhood of v is the set N[v] = N(v) ∪ {v}. For a set S ⊂ V, the open neighbourhood of S is N(S) = $\bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is N[S] = N (S) ∪ S. A set S ⊂ V is a vertex covering of G if every edge uv ϵ E is adjacent to at least one vertex in S. The vertex covering number β(G) is the minimum cardinality of the vertex covering sets in G. A vertex covering set with cardinality β (G) is called a β- set. let c (G , i) be the family of vertex covering sets of G with cardinality i and let C(G , i) = |C(G, i)|, the polynomial, $C(G, x) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)\, x^i$ is defined as the vertex cover polynomial of G. In [3], many properties of the vertex cover polynomials have been studied.

Definition: 1.1

A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.

ISSN PRINT 2319 1775 Online 2320 7876
Research Paper UGC CARE Listed (Group-I) Journal Volume 11, Iss 10, Oct 2022

If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to join u and v ; the vertices u and v are called the end of e.

Definition: 1.2

The degree of a vertex v in G is the number of edges incident on it.

A graph G is said to be k-regular if all its vertices are of degree k.

Every pair of its vertices are adjacent in G, is said to be complete, the complete graph on 'n' vertices is denoted by K_n .

Definition: 1.3

Let C (G, i) be the family of all vertex covering sets of G with cardinality i and let

c(G, i) = |C (G, i)|. The vertex cover polynomial of G is defined as

$$C(G, x) = \sum_{i = \beta(G)}^{|v(G)|} c(G, i)\, x^{i}.$$

Theorem: 1

Let $G = K_{1,n} \times P_2$ be any graph of order $2(n+1)$, then

$$C(G,x) = \sum_{j=0}^{n+1}\left[\sum_{i=0}^{n+1-j} \{(n+1)C_i\}\{n-(i-1)\}C_j\right] x^{n+1+j}$$

Proof:

Given $K_{1,n}$ be any complete bipartite graph of order $(1,n)$ and P_2 is a complete graph of order 2. Let $G = K_{1,n} \times P_2$ then

$V(G) = \{u_i, v_i / 0 \le i \le n\}$ with $d(u_0) = d(v_0) = n+1$

and $d(u_i) = d(v_i) = 2$ for all $1 \le i \le n$.

Clearly$N(u_0) = \{v_o, u_i/1 \le i \le n\}$; $N(v_0) = \{u_o, v_i/1 \le i \le n\}$;

$N(u_i) = \{u_o, v_i\}$ and $N(v_i) = \{v_o, u_i\}$ for all $1 \le i \le n$

Choose S_1 and S_2 are the sub sets of $V(G)$ Such that

$S_1 = \{u_i/0 \le i \le n\}$ and $S_2 = \{v_i/0 \le i \le n\}$

Since every $u_i v_i \in E(G)$ for all $1 \le i \le n$ either u_i or $v_i \in V(G)$ is an element in any covering set of G. In similar $u_0 v_0 \in E(G)$. Hence either u_0 or v_0 belongs to all covering set of G. Therefore, the covering sets with minimum cardinality of G is $n+1$.

ISSN PRINT 2319 1775 Online 2320 7876
Research Paper UGC CARE Listed (Group-I) Journal Volume 11, Iss 10, Oct 2022

That is

$$C(G,n+1) = \{\, S_1; \{S_1 - \{u_i\} \cup \{v_i\}/0 \le i \le n\}; \{S_1 - \{u_i,u_j\} \cup \{v_i,v_j\}/0 \le i,j \le n \text{ and } i \ne j\};$$

$$\{S_1 - \{u_i,u_j,u_k\} \cup \{v_i,v_j,v_k\}/0 \le i,j,k \text{ and } i \ne j \ne k\}; \dots;$$

$$\{S_2 - \{v_i,v_j,v_k\} \cup \{u_i,u_j,u_k\}/0 \le i,j,k \text{ and } i \ne j \ne k\};$$

$$\{S_2 - \{v_i,v_j\} \cup \{u_i,u_j\} 0 \le i,j \le n \text{ and } i \ne j\}; ; \{S_2 - \{v_i\} \cup \{u_i\}/0 \le i \le n\}; S_2\}$$

$$|S_1| = |S_2| = n+1$$

Clearly, $|S_1 - \{u_i\} \cup \{v_i\} \,/\, 0 \le i \le n| = (n+1)$

$$|S_1 - \{u_i,u_j\} \cup \{v_i,v_j\} \,/\, 0 \le i,j \le n| = (n+1)C_2$$

$$|S_1 - \{u_i,u_j,u_k\} \cup \{v_i,v_j,v_k\}/0 \le i \le n| = (n+1)C_3$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

In similar,

$|S_2 - \{v_i\} \cup \{u_i\} \,/\, 0 \le i \le n| = (n+1)C_n$ and

$|S_2| = (n+1)C_{n+1}$

Hence, $c(G,n+1)$

$= 1 + (n+1)C_1 + (n+1)C_2 + \cdots + (n+1)C_n$

Therefore, $c(G,n+1) = 2^{n+1}$

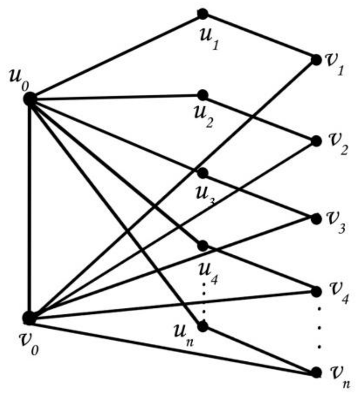

Figure:1

Covering sets with cardinality $n+2$ are

$C(G,n+2) =$

$\{\{elements\ of\ S_1\ with\ any\ one\ elements\ of\ S_2\}; \{S_1 - \{u_i\} \cup \{v_i,v_j\}/0 \le i,j \le n\ and\ i{\ne}j$

$$\left\{\begin{matrix} removal\ of\ any\ two\ elements\ (u_i,u_j)\ from\ S_1\ and\ the\ corresponding\ pair\ (v_i,v_j) \\ together\ with\ another\ elements\ v_k \in S_2/0 \le i,j,k \le n\ and\ i \ne j \ne k \end{matrix}\right\}$$

$$\{S_1 - \{any\ three\ elements\ u_i,u_j,u_k\} \cup \{v_i,v_j,v_k,v_l\}/0 \le i,j,k,l \le n\ and\ i \ne j \ne k \ne l\};$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$\{S_2 - \{v_i\} \cup \{u_i,u_j\}/0 \le i,j, \le n\ and\ i \ne j\}; \{S_2 \cup \{u_i\}/0 \le i \le n\,\};\ \}$$

ISSN PRINT 2319 1775 Online 2320 7876

Research Paper UGC CARE Listed (Group-I) Journal Volume 11, Iss 10, Oct 2022

Hence, the number of covering sets of G with cardinality $n+2$ is

$$|C(G,n+2)| = \{(n+1)C_0\}\{(n+1)C_1\} + \{(n+1)C_1\}\{nC_1\} + \{(n+1)C_2\}\{(n-1)C_1\} + \{(n+1)C_3\}\{(n-2)C_2\} \ldots\ldots\ldots\ldots\ldots\ldots..\{(n+1)C_{n-1}\}\{[n-(n-2)]C_1\} + \{(n+1)C_n\}\{[n-(n-1)]C_1\}$$

$$\Rightarrow |C(G,n+2)| = \sum_{i=0}^{n}\{(n+1)C_i\}\{n-(i-1)\}C_1$$

Covering sets of G with cardinality $n+3$ are

$$C(G,n+3) = \{\, elements\ of\ S_1\ with\ any\ two\ elemets\ of\ S_2;$$
$$\{S_1 - \{u_i\} \cup \{v_i, v_j, v_k\}/0 \le i,j,k \le n\ and\ v_i, v_j, v_k\ are\ distinct\}$$
$$\{S_1 - \{u_i, u_j\} \cup \{v_i, v_j, v_k, v_l\}/0 \le i,j,k,l \le n\ and\ i \ne j \ne k \ne l\}\,\}$$

In similar,

$$\left\{\begin{matrix} removal\ of\ any\ three\ elements\ from\ S_1\ with\ the\ corresponding\ three\ elements\ of\ S_2\ \ and \\ any\ two\ more\ elements\ among\ the\ remaining\ (n-2)\ elements\ of\ S_2\ which\ are \\ not\ already\ selected \end{matrix}\right\};$$

$$\ldots; \qquad \{S_2 - \{v_i, v_j\} \cup \{u_i, u_j, u_k, u_l\}/0 \le i,j,k,l \le n\ and\ i \ne j \ne k \ne l\};$$

$$\{S_2 - \{v_i\} \cup \{u_i, u_j, u_k\}/0 \le i,j,k \le n\ and\ \ i \ne j \ne k\}; \left\{S_2 \cup \ \{\{u_i\}/0 \le i \le n\}\right\}$$

Therefore,

$$c(G,n+3) = \{(n+1)C_0\}\{(n+1)C_2\} + \{(n+1)C_1\}nC_2 + \{(n+1)C_2\}(n-1)C_2 + \{(n+1)C_3\}\{(n-2)C_2\} + \cdots + \{(n+1)C_{n-2}\}.\{n-(n-3)\}C_2 + \{(n+1)C_{n-2}\}\{n-(n-2)\}C_2$$

Hence, $\quad c(G,n+3) = \sum_{i=0}^{n-1}\{(n+1)C_i\}\{n-(i-1)\}C_2$

In similar, $\quad C(G,n+4) = \sum_{i=0}^{n-2}\{(n+1)C_i\}\{n-(i-1)\}C_3$

Proceeding this way covering sets with cardinality $2n$ are

$$C(G,2n) = \Big\{\{The\ elements\ of\ S_1\ with\ any\ (n-1)\ elements\ of\ S_2\}; \{S_1 - \{u_i\} \cup \{v_i\}\ with\ any\ (n-1)\ elements\ of\ S_2\ other\ than\ \{v_i\}\}; \{S_1 - \{u_i, u_j\} \cup S_2\Big\}$$

Therefore,

ISSN PRINT 2319 1775 Online 2320 7876
Research Paper UGC CARE Listed (Group-I) Journal Volume 11, Iss 10, Oct 2022

$$c(G,2n) = \{(n+1)C_0\}\{(n+1)C_{n-1}\} + \{(n+1)C_1\}\{nC_{n-1}\} + \{(n+1)C_2\}\{(n-1)C_{n-1}\}$$

That is $c(G,2n) = \sum_{i=0}^{2}\{(n+1)C_i\}\{n-(i-1)\}C_{n-1}$

Covering sets with cardinality $2n+1$ are the elements of $V(G)$ except any one element

That is $C(G,2n+1) = \{\{S_1 \cup S_2 - \{u_i\} \,/\, 0 \le i \le n\}; \{S_1 \cup S_2 - \{v_i\} \,/\, 0 \le i \le n\}\}$

That is $c(G,2n+1) = (n+1)C_1 + (n+1)C_1$

$$= \sum_{i=0}^{1}\{(n+1)C_i\}\{n-(i-1)\}C_n$$

Finally the only covering set of G with cardinality of $2n+2$ in the elements of $V(G)$

ie $C(G,2n+2) = S_1 \cup S_2 \Rightarrow C(G,2n+2) = 1$

Hence, the covering set polynomial of G is

$$C(G,x) = \left[\sum_{i=0}^{n+1}\{(n+1)C_i\}\{n-(i-1)\}C_0\right]x^{n+1} +$$

$$\left[\sum_{i=0}^{n}\{(n+1)C_i\}\{n-(i-1)\}C_1\right]x^{n+2} +$$

$$\left[\sum_{i=0}^{n-1}\{(n+1)C_i\}\{n-(i-1)\}C_2\right]x^{n+3} +$$

$$\left[\sum_{i=0}^{n-2}\{(n+1)C_i\}\{n-(i-1)\}C_3\right]x^{n+4} +$$

$$\left[\sum_{i=0}^{n-(n-2)}\{(n+1)C_i\}\{n-(i-1)\}C_{n-1}\right]x^{2n} +$$

$$\left[\sum_{i=0}^{n-(n-1)}\{(n+1)C_i\}\{n-(i-1)\}C_n\right]x^{2n+1} +$$

$$\left[\sum_{i=0}^{n-n}\{(n+1)C_i\}\{n-(i-1)\}C_{n+1}\right]x^{2n}$$

ISSN PRINT 2319 1775 Online 2320 7876
Research Paper UGC CARE Listed (Group-I) Journal Volume 11, Iss 10, Oct 2022

Hence,

$$C(G,x)=\sum_{j=0}^{n+1}\left[\sum_{i=0}^{n+1-j}\{(n+1)C_i\}\{n-(i-1)\}C_j\right]x^{n+1+j}$$

Hence the Proof.

Theorem: 2

The coefficient of the cover polynomial of the graph $G=K_{1,n}\times P_2$ satisfies the property of log-concave.

Proof:

By theorem:1

$$C(G,x)=\sum_{j=0}^{n+1}\left[\sum_{i=0}^{n+i-j}\{(n+1)C_i\}\{n-(i-1)\}C_j\right]x^{n+1-j}$$

Hence,

$a_0=\sum_{i=0}^{n+1}\{(n+1)C_i\}\{(n-(i-1))C_0\};\quad a_1=\sum_{i=0}^{n}\{(n+1)C_i\}\{(n-(i-1))C_1\}$

$a_2=\sum_{i=0}^{n-1}\{(n+1)C_i\}\{(n-(i-1))C_2\};\quad a_3=\sum_{i=0}^{n-2}\{(n+1)C_i\}\{(n-(i-1))C_3\}$

… … … … … … … …

$$a_{n-1}=\sum_{i=0}^{2}\{(n+1)C_i\}\{(n-(i-1))C_{n-1}\};\quad a_n=\sum_{i=0}^{1}\{(n+1)C_i\}\{(n-(i-1))C_n\}$$

$$a_{n+1}=\sum_{i=0}^{0}\{(n+1)C_i\}\{(n-(i-1))C_{n+1}\}$$

Clearly, every $a_i^2\geq a_{i-1}.a_{i+1}$ for all $0<i<n$. Hence, coefficient of $C(G,x)$ satisfies the property of log-concave.

Results:

$C(G,x)$ is a covering polynomial of G where

$G=K_{1,n}\times P_2$ then

(i) $c(G,n+1)=2^{n+1}$

ISSN PRINT 2319 1775 Online 2320 7876
Research Paper UGC CARE Listed (Group-I) Journal Volume 11, Iss 10, Oct 2022

(ii) $c(G, 2n+2) = C(G, 2n) + 2[C(G, 2n+1)]$

(iii) $c(G, 2n) = 2n(n+1)$

(iv) $c(G, 2n+1) = 2n+2$.

The coefficient of the covering polynomial of $G = K_{1,n} \times P_2$ where $3 \leq n \leq 8$ is given as below.

G	$c(G,4)$	$c(G,5)$	$c(G,6)$	$c(G,7)$	$c(G,8)$	$c(G,9)$	$c(G,10)$	$c(G,11)$	$c(G,12)$	$c(G,13)$	$c(G,14)$	$c(G,15)$	$c(G,16)$	$c(G,17)$	$c(G,18)$
$K_{1,3} \times P_2$	16	32	24	8	1										
$K_{1,4} \times P_2$		32	80	80	40	10	1								
$K_{1,5} \times P_2$			64	192	246	160	60	12	1						
$K_{1,6} \times P_2$				128	448	672	560	280	84	14	1				
$K_{1,7} \times P_2$					256	1024	1792	1792	1120	448	112	16	1		
$K_{1,8} \times P_2$						512	2304	5608	5376	4032	2013	672	144	18	1

***References*:**

[1] *Alikhani. S and Peng. Y.H. Introduction to Domination Polynomial of a Graph. Ar.Xiv : 09052241 v1 [math.co] 14 May 2009.*

[2] *Alikhani. S and Peng. Y.H. Domination Sets and Domination polynomials of cycles. Global Journal of Pure and Applied Mathematics, Vol. 4, No. 2, 2008.*

[3] *Dong. F.M, Hendy M.D, Teo K.L. Little. C.H.C. The vertex – cover polynomial of a graph, Discrete Mathematics 250 (2002) 71 – 78.*

[4] *Douglas B. West, Introduction to Graph Theory.*

[5] *Frucht. R and Harary. F, Corona of two graphs, A equations. Math.4 (1970) 322-324.*

[6] *Gary Chartrand and Ping Zhang ; Introduction to Graph Theory.*

[7] *A. Vijayan , B. Stephen John,"On Vertex-Cover Polynomials on some standard Graphs" Global Journal of Mathematical Sciences: Theory and Practical's, 2012, ISSN No. 0974 – 3200, Volume 4, Number 2, pp 177-193.*

[8] *A. Vijayan , B. Stephen John "On the Coefficient of Vertex-Cover Polynomials of*

ISSN PRINT 2319 1775 Online 2320 7876
Research Paper UGC CARE Listed (Group-I) Journal Volume 11, Iss 10, Oct 2022

Paths" International Journal of Mathematical Sciences and Applications, 2012, ISSN No. 2230 – 9888, Volume 2, Number 2, pp 509-516.

[9] *A. Vijayan , B. Stephen John" On the Coefficients of Vertex-Cover Polynomials of Cycles"Advantages and Applications in Discrete Mathematics, 2012, Volume 10, Number 1, 2012, pp 23-38.*