INVERTIBLE PHASES AND FERMIONIC CRYSTALLINE EQUIVALENCE PRINCIPLE

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ABSTRACT: The classification of topological phases of matter has been the subject of intensive research in condensedmatter physics and nearby areas of mathematics for the last decade, but difficult problems still remain: for example, there is not yet an accepted mathematical definition of a topological phase of matter, so researchers must study these systems using ansatzes or heuristic definitions of phases. Restricting to invertible phases, also known as symmetry-protected topological (SPT) phases, simplifies the classification question, but defining these phases precisely is also still an open problem. This paper presents Invertible Phases and Fermionic Crystalline Equivalence Principle. We provide a model for the classification of invertible phases on a G-space. The aim of this paper is to formulate and prove such a fermionic crystalline equivalence principle (FCEP). To do so, we provide an ansatz expressing groups of invertible phases on a G-space Y in which the symmetry type can be merely locally constant over space and can mix with G, including as a special case spatial symmetries mixing with fermion parity. The computation of phase homology groups that represent groups of pointgroup-equivariant fermionic phases reduces to computations of bordism groups. For symmetry types, that has been studied before by other methods, our computations agree with the literature, bolstering our ansatz.

KEYWORDS: symmetry-protected topological (SPT), fermionic crystalline equivalence principle (FCEP).

I. INTRODUCTION

Topological phases of matter:

Physicists studying condensed-matter systems discovered that some of them have unusual properties; for example, these systems can have “quasiparticles,” localized excitations that behave like particles but are not actually particles, instead consequences of the highly entangled nature of the electrons in the system [1]. These quasiparticles can have unusual statistics: for example, some of these systems are lower-dimensional and can therefore have quasiparticles which are neither bosons nor fermions. These systems are examples of topological phases of

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matter — a topological phase is something like an equivalence class of systems with the same physical properties, though a precise definition is not yet available [2].

When studying a topological phase of matter, it is important to specify a collection of symmetries that should not be broken. For example, some topological phases have a time-reversal symmetry, and others do not [3]. The classifications of topological phases with time-reversal symmetry and without it are expected to be different — there can be ways to deform one system into another that break time-reversal symmetry, so these phases are inequivalent as phases with time-reversal symmetry but equivalent in its absence. Specifying the collection of symmetries is akin to specifying the symmetry type of a TFT [4].

Heuristic Definition 1:
A symmetry-protected topological (SPT) phase, or a short-range entangled (SRE) phase, or an invertible phase, is a topological phase of matter which is invertible under stacking, i.e. there is some other topological phase with the same symmetries and such that when those two phases are stacked, the resulting system is in the trivial phase [5]. Under stacking, SPTs for a given dimension and collection of symmetries, SPTs form an abelian group.

Heuristic Definition 2:
A topological phase of matter in which the collection of symmetries acts nontrivially on space is called a crystalline topological phase or crystalline SET phase. If it is invertible, it is called a crystalline SPT phase.

A Hamiltonian formalism for topological phases of matter:
Physicists study topological phases of matter from several different viewpoints. We will focus on the lattice Hamiltonian formalism: studying a phase by discretizing space and writing down a state space and Hamiltonian using that discrete data. This is a common approach to modeling topological phases. A lattice Hamiltonian model for a topological phase is roughly a procedure for assigning to a manifold with a triangulation the data of a complex Hilbert space H, called the state space, and a self-adjoint operator H : $\mathcal{H} \to \mathcal{H}$ called the Hamiltonian. Sometimes the combinatorial structure is more general or less general than a triangulation; sometimes the manifold has additional structure encoded combinatorially, such as a spin structure or principal bundle, and the state space and Hamiltonian may depend on this data.

In the Hamiltonian formalism, the symmetries of the phase are encoded in this discrete data: for example, crystalline phases with a C_4 rotation symmetry can be modeled on a square lattice with a Hamiltonian that is invariant under $\pi/2$ rotations. Understanding how to express these symmetries using the Hamiltonian is understood in many cases but not in complete generality. Two Hamiltonian lattice models which can be defined using the same triangulation can be

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stacked: say their state spaces are H1 and H2 and their Hamiltonians are H1: $\mathcal{H}_1 \rightarrow \mathcal{H}_1$ and H2 : $\mathcal{H}_2 \rightarrow \mathcal{H}_2$. Stacking these two systems means considering the system with state space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and Hamiltonian.

$$\mathrm{H}_1 \otimes 1 + 1 \otimes \mathrm{H}_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \ldots (1)$$

The low-energy ansatz includes a correspondence between the collections of symmetries as expressed in physics and symmetry types of TFTs: for example, fermionic phases with a time-reversal symmetry squaring to 1 are believed to correspond to pin− topological field theories, in the sense that the low-energy limit of such a phase is expected to be a pin− TFT. Under this ansatz, stacking corresponds to the tensor product of TFTs. Therefore the low-energy limit of an SPT (ignoring crystalline symmetries for now) should be a reflection positive invertible field theory, and the corresponding classifications should match.

Freed-Hopkins formulate an ansatz for invertible phases of matter on a topological space Y equipped with an action of a compact Lie group G. First, specify the symmetry type of the theory as a map ρ: H → O, where $O := \underset{\rightarrow n}{Lim} O_n$ is the infinite orthogonal group and H is a topological group. For fermionic phases, Ansatz is not the full answer, and providing the full answer is a major goal of this paper. Physicists distinguish between phases with "spinless fermions" and "spin-1/2 fermions", asking how the spatial symmetry group G mixes with fermion parity.

Kitaev proposed that the classification of invertible phases on a space Y should form a generalized homology theory, and Freed-Hopkins use this to make an ansatz computing groups of phases on G-spaces as Borel-equivariant generalized homology groups. We extend Freed-Hopkins' ansatz to the case where the G-symmetry on space can mix with the internal symmetry of the phase, so as to account for, for example, fermionic phases with a C_4 rotation symmetry, such that a full 2π rotation acts on fermions by −1. Such mixed symmetries have been studied in the physics literature where G is a group of rotations, reflections, inversions, or rotations and reflections.

We next address the fermionic crystalline equivalence principle. Crystalline equivalence principles express the classification of crystalline phases, which are certain topological phases of matter in which a group of symmetries acts on space, in terms of classifications of phases without such a spatial symmetry. The equivalence principle for bosonic SPT phases (corresponding to symmetry types such as O or SO) is well-understood, and for fermionic crystalline SPTs (corresponding to Spin, Pin±, etc.) it is understood in special cases. We provide a general fermionic crystalline equivalence principle (FCEP) in Altland-Zirnbauer classes D and A, corresponding to symmetry types $\mathrm{H} = \mathrm{Spin}$ and $\mathrm{H} = \mathrm{Spin}^{\mathrm{C}}$. The theorem identifies phase

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homology groups with groups of reflection positive invertible field theories, but with a twist: the symmetry types on the two sides of this equivalence do not match, and are exchanged.

II. PHASES ON A G-SPACE

We reprise the ansatz of Freed-Hopkins on invertible phases on a G-space, though we need to generalize it: physicists often consider crystalline phases in which the symmetry acting on spacetime mixes with the internal symmetry (e.g. a reflection squaring to $(-1)^{F}$, leading us to generalize from homology to twisted homology.

Invertible phases on a space:

Let Y be a locally compact topological space and C an ∞- category. Following Ando-Blumberg-Gepner, we say a C-valued local system on Y is a functor $\mathcal{L}: \pi \leq \infty \, Y \to C$ here $\pi \leq \infty \, Y$ is the fundamental ∞-groupoid of Y. If $\mathcal{L}: Y \to S_P$ is a local system of spectra, the homology of Y valued in $\mathcal{L}$ is $\mathcal{L}_*(Y) \coloneqq \pi_*$ (hocolim $\mathcal{L}$), and the cohomology of Y valued in $\mathcal{L}$ is $\mathcal{L}^*(Y) \coloneqq \pi_*$ (holim $\mathcal{L}$); this generalizes (co)homology with local coefficients.

Given a subspace j: $Y^1, \to Y$, we also define relative homology groups: j induces a map $j_*: \underset{Y^|}{\text{hocolim}}\, \mathcal{L}|Y^| \to \underset{Y}{\text{hocolim}}\, \mathcal{L}$ and we define $\mathcal{L}(Y, Y^|) \coloneqq \pi_*(\text{cofib}(j_*))$.Relative cohomology is analogous.

Invertible phases on a G-space:

Our model for invertible crystalline phases requires considering the case where a compact Lie group G acts on Y. Again we closely follow Freed-Hopkins but using twisted Borel-Moore homology. G is a Lie group, we do not need G to be compact. Indeed, in the study of crystalline phases, G is often an infinite discrete subgroup of $Isom\ (E^n)$, We work with the ∞-category S_P^G of Borel G-equivariant spectra, whose objects can be modeled by data of a sequence of G-spaces X_n together with G-equivariant maps $\sum X_n \to X_{n+1}$. Notions of homotopy equivalence, etc., and do not require their compactness assumption on G.

III. THE FERMIONIC CRYSTALLINE EQUIVALENCE PRINCIPLE

This section goal is to state and prove the FCEP (Fermionic Crystalline Equivalence Principle), identifying phase homology groups in classes D and A with groups of deformation classes of invertible field theories. Assuming Ansatz, this leads to the more familiar version of the FCEP: crystalline equivalence principles are first introduced by Thorngren-Else: the idea is to equate the classification of crystalline topological phases of matter for some group G acting on spacetime with a classification of a different kind of topological phases of matter, in which G is part of the internal symmetry group. Then one may use preexisting techniques for phases without a spatial symmetry to classify phases with the specified G-action on space.

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The fermionic analogue of this statement is more complicated because there are more ways for G to mix with the symmetry type. Previous study examples in which an FCEP holds, and each paper discusses that such a principle would have to account for the different ways in which G mixes with H: crystalline phases for which the spatial G-symmetry does not mix with fermion parity correspond to phases with an internal G-symmetry that does mix with fermion parity, and vice versa. Examples of this twisted correspondence also appear in work of Freed-Hopkins.

Our version of the FCEP applies in Altland-Zirnbauer classes A and D (i.e. $H = Spin\,, H = Spin^{C}$), for all compact Lie groups G acting on faithfully on space, and all ways in which G may mix with fermion parity. The slogan "mixed crystalline goes to unmixed internal, and vice versa" is a little hard to glean from the result when the G-action includes reflections, but we obtain an equivalence from phase homology groups for certain equivariant local systems of symmetry types, which under Ansatz stands in for groups of crystalline SPT phases, to groups of deformation classes of IFTs, which under Freed-Hopkins' ansatz model groups of phases without spatial symmetries.

Theorem (Fermionic crystalline equivalence principle):

Fixing data of G, H, λ, etc. as above, let f0, $f_{1/2}$denote the local systems of symmetry types for the case of spinless, resp. spin-1/2 fermions. Then $Ph_0^G(R^d; f_0)$ is isomorphic to the group of deformation classes of d-dimensional IFTs for the spin-1/2 internal symmetry type, and $Ph_0^G(R^d; f_{1/2})$ is isomorphic to the group of deformation classes of d-dimensional IFTs for the spinless internal symmetry type.

$$Ph_0^G(R^d; f_0) \xrightarrow{\cong} \left[MT_\rho\left(\frac{1}{2}\right), \Sigma^{d+k+2} I_Z\right] \dots (2)$$

$$Ph_0^G\left(R^d; f_{\frac{1}{2}}\right) \xrightarrow{\cong} \left[MT_\rho(0), \Sigma^{d+k+2} I_Z\right] \dots (3)$$

Assuming Ansatz, the physics implication of this theorem is that the abelian group of crystalline SPT phases for the spinless equivariant local system of symmetry types is naturally isomorphic to the abelian group of deformation classes of IFTs for the spin-1/2 internal symmetry type; and the classification of crystalline SPT phases for the spin-1/2 equivariant local system of symmetry types is naturally isomorphic to the abelian group of deformation classes of IFTs of the spinless internal symmetry type.

Proof:

First, Proposition simplifies the question into one of ordinary stable homotopy theory. We obtain Thom spectra for vector bundles over BHe, and to finish we must compare these spectra to

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$MTH\wedge(BG)^{E}$ where E → BG is some rank-zero virtual vector bundle. This comparison, in the form of shearing arguments, is the core of the proof. And after that proving Theorem simplifying the crystalline symmetry types match the Thom spectra for the internal symmetry types in Definitions.

The proofs of Theorem resemble the proofs of the more standard equivalences,

$$MTPin^{+} \cong MTSpin \wedge (BZ/2)^{1-\sigma} \ldots (4)$$
$$MTPin^{-} \cong MTSpin \wedge (BZ/2)^{\sigma-1} \ldots (5)$$
$$MTPin^{C} \cong MTSpin^{C} \wedge (BZ/2)^{\pm(1-\sigma)} \ldots (6)$$
$$MTSpin^{C} \cong MTSpin \wedge (BSO_2)^{\pm(2-V_2)} \ldots (7)$$

Where, $\sigma \rightarrow BZ/2$ and $V_2 \rightarrow BSO_2$ denote the respective tautological line bundles. These decompositions were first proven by Kirby-Taylor, Lemma 6] (Pin^{+}), Peterson (Pin^{-}), and BahriGilkey ($Spin^{C}$ and Pin^{C}). For a unified proof of all of these equivalences, see FreedHopkins.

IV. COMPUTATIONS

We study the fermionic crystalline equivalence principle in many examples where the symmetry is given by a two- or three-dimensional point group. We first study some cases already present in the literature and find agreement, including reflections in Altland-Zirnbauer classes D and A, inversions in classes D and A cyclic groups acting by rotations in classes D and A, and dihedral groups acting by rotations and reflections in class D. In all cases we consider both spinless and spin-1/2 fermions.

Then we calculate equivariant phase homology groups on R^3 for A_4 acting by tetrahedral symmetry and find that for classes A and D and the spinless and spin-1/2 cases, the zeroth phase homology groups all vanish. Under our ansatz, this predicts there are no nontrivial fermionic phase's equivariant for a tetrahedral symmetry in these cases. We predict that for 3d class D phases with a full tetrahedral symmetry (i.e. including reflections) and spinless fermions, there is a phase generating a Z/4 subgroup. This phase homology calculation required the most involved mathematical argument, and it would be interesting to see a physical description.

Our computations predict plenty of other phases, but many of them either have Adams filtration zero and therefore are not predicted to be intrinsically fermionic, or have more complicated symmetry types, such a full octahedral symmetry, that would be harder to study on the lattice.

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V. CONCLUSION

In this paper, Invertible Phases and Fermionic Crystalline Equivalence Principle is described. We provide a model for the classification of invertible phases on a G-space. This paper is formulated and proved such a fermionic crystalline equivalence principle (FCEP). Therefore, we provide an ansatz expressing groups of invertible phases on a G-space Y in which the symmetry type can be merely locally constant over space and can mix with G, including as a special case spatial symmetries mixing with fermion parity. The computation of phase homology groups that represent groups of pointgroup-equivariant fermionic phases reduces to computations of bordism groups. For symmetry types, that has been studied before by other methods, our computations agree with the literature, bolstering our ansatz.

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