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Chaotic Dynamics of a Continuous Innovation Diffusion Model with Delay and External as well as Internal Influences

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Abstract:

In the article, a model of diffusion of a product with three classes of populations viz. non-adopters from low-income group $N_1(t)$, non-adopters from high-income group $N_2(t)$ and adopters of the innovation A(t) is proposed to examine the qualitative behavior of an innovation in a region. It is noticed in the model that the increase in adopter's population is completely affected by the external (advertisements) as well as internal influences (word of mouth communications). Bifurcation in an innovation diffusion model is a phenomenon in which the model possesses stable interior equilibrium together with a stable adopter-free equilibrium. This model established two equilibriums, viz., the adopter-free steady state and an interior steady state. Dynamical behavior and the basic influence number of the proposed system are studied. The presence and local stability of the adopter-free and interior steady state are examined in terms of the effective Basic Influence Number (BIN) (R_A). It is investigated that the adopter free point is locally stable if $R_A < 1$. In this work, the author investigates the possibility of bifurcation in an innovation diffusion model. By considering τ (evaluation time period) as a control parameter, the system has been studied by classical theory of stability and Hopf-bifurcation analysis. The author has been able to investigate that

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the obtained critical value of parameter τ is responsible for the occurrence of periodic solutions due to Hopf bifurcation. The author believes that the presented research work in this chapter provides an insight of the dynamics of a diffusion model and possible causes of bifurcation in the same. Exhaustive numerical techniques have been applied in the justification of analytical results.

Key Words: Nonlinear Mathematical System, Basic Influence Number, Stability, Hopf-bifurcation, Sensitivity Analysis.

1. Introduction

Innovation diffusion is a never-ending process that completely depends upon a network and the diffusion of the innovation. It means that the idea that which transmits information between the populations of a socially arranged channel with the passage of time [1, 2]. Although the Roger's model of product and its diffusion was used and accepted in markets yet it has many shortcomings. The shortcomings of Roger's system was firstly studied and analyzed by Bass in the Bass model (1969) [3], and given a totally new model of innovation theory in relative of Rogers's model. The Bass model applied to understand and estimate the growth patterns of adopters by the impact of external influences such as mass media including print and electronic coverage, and internal communications and extensive influences [4, 5, 6, 7, 8, 9, 10]. The Bass model involving all the above-mentioned external as well as internal factors developed by Bass is given as below:

$$\frac{d}{dt}[A(t)]=(m-A(t))\left(p+\frac{q}{m}A(t)\right),$$

In article [11], the information stage and then a final adoption stage have been used into the Bass model. In Bass model, it is supposed that the number of potential-adopters remains completely unchanged in the innovation process but does not resembles with real situations,

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where these changes in the distinct demographic processes of the population may be changed by emigration, immigration and intrinsic growth [12, 13].

The analyses of Cellular Automata have been thoroughly represented by internal relations: a standard edge between two different agents or a reflexive edge if we refer to a single agent taking into account "auto-communication" (Guseo and Guidolin [14]). Mahajan and Sharma [15] have accounted the drawbacks of Bass model, and have advised a simple procedure for the estimation of diffusion of a product. Another alternative technique have been applied for the estimation of diffusion is the Augmented Kalman Filter (Xie et.al, [16]). Kennedy [17] focused on the problematic aspect of the 'managerial intuition' which is not correctly estimated usually in the model. The term managerial intuition refers to extraneous various sources that specify the parameter values. For the spread of contamination with an association of infected individual to another populations is investigated by authors in [18]. Thus, in similar manner, the non-adopter may become the adopter of a product by the help of internal factors [19]. The short-term diffusion of electronic industries in Spain was studied in [20]. In the communication market, the researchers have investigated the dynamics of competitive multi-innovations and its application [21, 22].

This paper is managed as: In following section, a diffusion model involving delay as control strategy, along with external and internal influences is formulated. In section 3, the basic preliminaries including the positivity and boundedness of model are justified. The next section related to discuss the concept of existence of equilibrium points and basic influence numbers and also guided us to deal with the dynamical analysis, and find out the local asymptotic stability conditions of interior equilibrium point. The model is investigated for the Hopf bifurcation analysis in the Section 5. In section 6, the numerical simulations are performed. In the result section, the main findings are presented with their significance to

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actual life scenarios.

1. Formulation of Mathematical model

The finance positions of the various populations may be changed with the passage of time. The given innovation diffusion system is formulated by applying the high-end products. The products, which are not everyone cup of tea, i.e., which not to be used by every individual in the society is taken into consideration. The population is sub-classified into three classes viz., the non-adopters population of low income group population density $N_1(t)$, the non-adopters of high income group population density $N_2(t)$, and adopter population with density $A(t)$, where t denotes the time. For making this model more realistic, and effective, the various demographic processes of a population have been taken into consideration into the traditional Bass model. The suppositions for the construction of the mathematical system are detailed and given below:

1. Let Λ_1 and Λ_2 be the new recruitment rates of non-adopter populations $N_1(t)$ and $N_2(t)$ respectively.

2. As we know, the financial stature of the population always changes with the change of time. It means that the non-adopter population of low-income group $N_1(t)$ can be the population of high income group $N_2(t)$ in future times and vice-versa. It clearly indicates that their populations can migrate to the groups of each other according to their current financial positions. Let us suppose that ξ_1 is the rate at which the non-adopter population of low-income group $N_1(t)$ joins the population of high-income group $N_2(t)$ and ξ_2 is the rate at which the non-adopter population of high-income group $N_2(t)$ joins the population of low income group $N_1(t)$.

3. As we are taking the high-end innovations, and so it is assumed that only the non-adopters

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population of high-income group $N_2(t)$ is available to use the product.

4. Let us assume that the variable external influences (advertisements) and variable internal influences (word of mouth) directly influence the decision about the adoption of a product.

5. Assume that α is the rate of mutual communications (word of mouth) occurred. Also let β be the parameter used for variable external influences (advertisements).

6. It is assumed that the shifting of non-adopters to the adopter population is not instantaneous, but takes small time defined by the average evaluation period τ. So $\left(\alpha A(t-\tau)+\beta\right)N(t-\tau)$ is the non-adopters directly transferring to adopter population having knowledge about the product at $t-\tau$ by various influences, who join the adopter class $A(t)$ at time t.

7. Let v_1 is the rate of adopters joining back to $N_1(t)$ and v_2 is the rate of adopters joining back to $N_2(t)$ because of getting dissatisfied by the performance of the product that of course may come again to join later depending upon all the circumstances. Suppose that δ is the death and emigration rate of the various populations $N_1(t)$, $N_2(t)$ and $A(t)$.

Here, in the following system (6.1), to observe the effect of τ, the author consider it as a control parameter. Thus, the mathematical form of the model is given as below:

$$
\begin{aligned}
\frac{dN_1}{dt} &= \Lambda_1 - \xi_1 N_1(t) + \xi_2 N_2(t) - \delta N_1(t) + v_1 A(t), \\
\frac{dN_2}{dt} &= \Lambda_2 + \xi_1 N_1(t) - \xi_2 N_2(t) - \left(\alpha A(t-\tau)+\beta\right) N_2(t-\tau) e^{-(\delta+\rho)\tau} + v_2 A(t) - \delta N_2(t), \\
\frac{dA}{dt} &= \left(\alpha A(t-\tau)+\beta\right) N_2(t-\tau) e^{-(\delta+\rho)\tau} - (\delta + v_1 + v_2) A(t), \qquad (6.1)
\end{aligned}
$$

These equations are to be solved subject to the initial conditions

$$N_1(\theta)=\phi_1(\theta),\ N_2(\theta)=\phi_2(\theta),\ A(\theta)=\phi_3(\theta);\ \phi_i(0)>0,\ \forall\ i=1,2,3 \qquad (6.2)$$

The functions $\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)$ are continuous functions, which are bounded in the interval $[-\tau, 0]$. Specifically, $\phi_1(\theta), \phi_2(\theta), \phi_3(\theta) \in C([-\tau, 0], \Re_+{}^3)$, the Banach space of functions mapping the interval $[-\tau, 0]$ into $\Re_+{}^3 = \{(y_1, y_2, y_3) : y_i > 0, i = 1, 2, 3\}$,which are continuous. Applying the basic results of FDE [23], make a note that all the solutions $N_1(\theta), N_2(\theta), A(\theta)$ of the IVP are always unique and nonnegative on [0, +∞).

2. Fundamental Preliminaries

Positive solutions of the model (1) should be found in the boundary of an area for a well-posed mathematical problem.

Lemma3.1. Corresponding to the initial values (2), all the solutions of (1) are positive $\forall t \geq 0$.

Proof. Considering $t \in [0, \tau]$, model (1) can be re-written as

$$\frac{dN_1}{dt} = \Lambda_1 + \xi_2 N_2 + v_1 A - (\xi_1 + \delta) N_1(t),$$

$$\Rightarrow N_1(t) \geq N_1(0) e^{-\int_0^t (\xi_1 + \delta) ds} > 0$$

The second equation of the system can also be written as:

$$\frac{dN_2}{dt} + (\xi_2 + \delta) N_2(t) \geq -(\alpha A(t-\tau) + \beta) N_2(t-\tau) e^{-(\delta+\rho)\tau}$$

On solving the equation for $t \in [0, \tau]$,

$$N_2(t) \geq e^{-\int_0^t (\xi_2+\delta) d\theta} [N_2(0) - \int_0^t \{(\alpha A(\theta-\tau) + \beta) N_2(\theta-\tau)\} e^{-(\delta+\rho)\tau} e^{\int_0^\theta (\xi_2+\delta) ds} d\theta] > 0$$

Now, the third Eq. of the model (1) provides us:

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$$\frac{dA}{dt}+(\delta+v_1+v_2)A(t)=\left(\alpha A(t-\tau)+\beta\right)N_2(t-\tau)e^{-(\delta+\rho)\tau},$$

$$A(t)=e^{-\int_0^t(\delta+v_1+v_2)d\theta}[A(0)+\int_0^t\{\left(\alpha A(\theta-\tau)+\beta\right)N_2(\theta-\tau)\}e^{-(\delta+\rho)\tau}e^{\int_0^\theta(\delta+v_1+v_2)ds}d\theta]$$
$$\geq 0,\text{ for } t\in[0,\tau].$$

Similarly, the above results can be obtained in the successive intervals $[\tau,2\tau],[2\tau,3\tau],.....,$

$....[n\tau,(n+1)\tau];\ n\in\mathbb{N}$.

Hence, we can conclude that all the solutions of the model (1) are nonnegative $\forall\ t\geq 0.$

Lemma 3.2. The non-negative solutions of (1) corresponding to the initial values (2) in $\Re_+{}^3$ are bounded.

Proof. Construct $\Upsilon(t)=N_1(t)+N_2(t)+A(t),\ \forall t\geq 0.$

Differentiating w.r.t. *t*, we get

$$\frac{d\Upsilon}{dt}+\delta\Upsilon=\Lambda_1+\Lambda_2,\ or\ \Upsilon e^{\delta t}=\int(\Lambda_1+\Lambda_2)e^{\delta t}dt+C",\ or\ \Upsilon=\frac{(\Lambda_1+\Lambda_2)}{\delta}+C"e^{-\delta t}$$

Using a significant result from differential inequalities and positivity of all the three classes, we get:

$$0\leq\Upsilon(t)=\frac{\Lambda_1+\Lambda_2}{\delta}(1-e^{-\delta t})+\Upsilon(0)e^{-\delta t},\text{ for any }\delta>0.$$

As $t\to\infty$, we have $0\leq\Upsilon(N_1,N_2,A)\leq\frac{\Lambda_1+\Lambda_2}{\delta}$.

Thus, all solutions of the model (1) which initiate in $\Re_+{}^3$ will stay in the region

$$\Im=\{(N_1(t),N_2(t),A(t)):0\leq\Upsilon(t)\leq\frac{\Lambda_1+\Lambda_2}{\delta}+\in,\text{ for any }\in>0\}.$$

Thus $N_1(t), N_2(t), A(t)$ will always be bounded above by the bounds lying in the region $\Im$, $\forall\ t \geq 0$.

4. Equilibrium points and stability analysis

4.1 Existence of Equilibria

To obtain the possible steady points of the model (1), it must satisfy the following equations:

$$\begin{aligned}&\Lambda_1 - \xi_1 N_1(t) + \xi_2 N_2(t) - \delta N_1(t) + v_1 A(t) = 0,\\ &\Lambda_2 + \xi_1 N_1(t) - \xi_2 N_2(t) - \left(\alpha A(t) + \beta\right) N_2(t) + v_2 A(t) - \delta N_2(t) = 0,\\ &\left(\alpha A(t) + \beta\right) N_2(t) - (\delta + v_1 + v_2) A(t) = 0.\end{aligned} \tag{3}$$

While solving the above equations, following possible equilibrium points has been obtained:

4.1.1Non-adopter-*N2(t)* and Adopter *A(t)* free equilibrium point

To get the above required point, $N_2(t) = 0$ and $A(t) = 0$ has been substituted in equation (3) which provides us: $\Lambda_1 - \xi_1 N_1 - \delta N_1 = 0$, $N_1 = \dfrac{\Lambda_1}{(\xi_1 + \delta)}$.

Thus, the required equilibrium point has been derived as $E^{(1)} = (N_1(t),\ 0,\ 0)$ where $N_1(t) = \dfrac{\Lambda_1}{\xi_1 + \delta}$ which is always feasible.

4.1.2 Adopter free equilibrium point

To get the adopter free steady state, substitute A(t) = 0 in the equations (3). Required point will be $E^{(2)} = (N_1(t), N_2(t),\ 0)$ where $N_1(t)$ and $N_2(t)$ must satisfy the equations:

$\Lambda_1 - \xi_1 N_1(t) + \xi_2 N_2(t) - \delta N_1(t) = 0$ and $\Lambda_2 + \xi_1 N_1(t) - \xi_2 N_2(t) - \beta N_2(t) - \delta N_2(t) = 0$.

These equations are re-written as: $(\xi_1 + \delta) N_1(t) - \xi_2 N_2(t) = \Lambda_1$

And $-\xi_1 N_1(t) + (\xi_2 + \beta + \delta) N_2(t) = \Lambda_2$.

Using the elimination method to solve the above equations, the adopter free feasible

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equilibrium point $E^{(2)} = (N_1(t), N_2(t), 0)$ has been obtained where:

$$N_1(t) = \frac{(\Lambda_1 + \Lambda_2)\xi_2 + \Lambda_1(\beta + \delta)}{\delta(\xi_2 + \beta + \delta) + \xi_1(\beta + \delta)}, N_2(t) = \frac{(\Lambda_1 + \Lambda_2)\xi_1 + \Lambda_2\delta}{\delta(\xi_2 + \beta + \delta) + \xi_1(\beta + \delta)}.$$

4.1.3 Basic influence number

Since the Adopter free steady state always exists, thus, there must be a thorough explanation of the potential for the existence of non-negative steady state. Before getting non negative point E^*, first we need to find the basic influence number which is similar to the basic reproduction amount of epidemiology. Basic influence number [24] describes the mathematical framework for the product diffusion. To find this number, we need to find two vectors ω_1 and ς representing new users sequentially based on the personal effects of users on the non-users populations and lasting conditions respectively.

For our model system the vectors ω_1 and ς are given as:

$$\omega_1 = \begin{bmatrix} \alpha N_2 A \\ 0 \\ 0 \end{bmatrix}, \varsigma = \begin{bmatrix} (\delta + v_1 + v_2)A - \beta N_2 \\ -\Lambda_2 - \xi_1 N_1(t) + \xi_2 N_2(t) + (\alpha A + \beta) N_2 - v_2 A + \delta N_2 \\ -\Lambda_1 + \xi_1 N_1 - \xi_2 N_2 + \delta N_1 - v_1 A \end{bmatrix}.$$

The Jacobian of the vectors ω_1 and ς at adopter free steady state $E^{(2)} = (N_1(t), N_2(t), 0)$ are obtained as Ω and Ξ respectively where:

$$\Omega = [\alpha N_2] \text{and } \Xi = [\delta + v_1 + v_2] \Rightarrow \Omega\Xi^{-1} = \left[\frac{\alpha N_2}{\delta + v_1 + v_2}\right].$$

Here, it is to be noticed that largest absolute eigen value of $\Omega\Xi^{-1}$ is same as the spectral radius of $\Omega\Xi^{-1}$, known as Basic influence number.

It's eigen value is obtained by solving $|\Omega\Xi^{-1} - \lambda I| = 0 \Rightarrow \lambda = \dfrac{\alpha N_2}{\delta + v_1 + v_2}$.

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Thus, the Basic influence number of the proposed model is

$$\Re_0 = \frac{\alpha N_2}{\delta + v_1 + v_2} = \frac{\alpha[(\Lambda_1 + \Lambda_2)\xi_1 + \Lambda_2\delta]}{(\delta + v_1 + v_2)[\delta(\xi_2 + \beta + \delta) + \xi_1(\beta + \delta)]}.$$

4.1.4 The non-trivial interior equilibrium point

To achieve the non-trivial interior equilibrium point $E^* = (N_1^*(t), N_2^*(t), A^*(t))$, it must satisfy the equations (3).

From the last equation of (3), we have $(\alpha A + \beta)N_2 - (\delta + v_1 + v_2)A = 0$

$$\Rightarrow N_2 = \frac{(\delta + v_1 + v_2)A}{(\alpha A + \beta)} \tag{4}$$

Using the value of N_2 in the first equation of (3), we have:

$$\Lambda_1 - (\xi_1 + \delta)N_1 + \xi_2\left[\frac{(\delta + v_1 + v_2)A}{(\alpha A + \beta)}\right] + v_1 A = 0,$$

$$\Rightarrow N_1 = \frac{(\Lambda_1 + v_1 A)(\alpha A + \beta) + (\delta + v_1 + v_2)\xi_2 A}{(\alpha A + \beta)(\xi_1 + \delta)}. \tag{5}$$

Substituting the values of N_1 and N_2 and using the last equation of (3) in the middle equation of (3), we get:

$$\Lambda_2 + \xi_1\frac{(\Lambda_1 + v_1 A)(\alpha A + \beta) + (\delta + v_1 + v_2)\xi_2 A}{(\alpha A + \beta)(\xi_1 + \delta)} - (\xi_2 + \delta)\frac{(\delta + v_1 + v_2)A}{(\alpha A + \beta)} - (\delta + v_1)A = 0$$

Simplifying this expression, a quadratic equation in A has been derived as:

$f(A) = f_1 A^2 + f_2 A + f_3 = 0$, where:

$$f_1 = -\alpha\delta(\xi_1 + \delta + v_1),$$
$$f_2 = (\alpha\Lambda_1 + \beta v_1)\xi_1 + (\alpha\Lambda_2 - \beta(\delta + v_1) - \delta(\delta + v_1 + v_2))(\xi_1 + \delta) - \xi_2\delta(\delta + v_1 + v_2),$$
$$f_3 = \beta\Lambda_2(\xi_1 + \delta) + \beta\Lambda_1\xi_1.$$

Here, f_1 is always negative and f_3 is always positive. So, by Descartes's method of sign, the

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above equation must have a unique positive root and the positive value of $A = A^*$ is obtained from the corresponding positive root of the quadratic equation.

Therefore, non-trivial interior equilibrium point $E^* = (N_1^*(t), N_2^*(t), A^*(t))$ has been accomplished using the value of A^* in equation (4) and (5).

4.2 Stability Analysis of various equilibria

4.2.1 Analysis of the model, i.e., $\tau = 0$

The Jacobian matrix of the model (1) in absence of delay is:

$$J = \begin{bmatrix} -\xi_1 - \delta & \xi_2 & v_1 \\ \xi_1 & -\xi_2 - \alpha A - \beta - \delta & -\alpha N_2 + v_2 \\ 0 & \alpha A + \beta & \alpha N_2 - (\delta + v_1 + v_2) \end{bmatrix} \quad (6)$$

Theorem.4.1 The point $E^{(1)} = (N_1(t), 0, 0)$ with $N_1(t) = \dfrac{\Lambda_1}{\xi_1 + \delta}$ is always Locally asymptotically stable.

Proof. The Jacobian matrix corresponding to the equilibria $E^{(1)} = (N_1(t), 0, 0)$ is:

$$J^{(1)} = \begin{bmatrix} -\xi_1 - \delta & \xi_2 & v_1 \\ \xi_1 & -\xi_2 - \beta - \delta & v_2 \\ 0 & \beta & -(\delta + v_1 + v_2) \end{bmatrix}$$

The characteristic equation of $J^{(1)}$ is $| J^{(1)} - \lambda \, I| = 0$ or

$$(-\delta - \lambda) \begin{vmatrix} 1 & 0 & 0 \\ \xi_1 & -\xi_1 - \xi_2 - \beta - \delta - \lambda & v_2 - \xi_1 \\ 0 & \beta & -(\delta + v_1 + v_2) - \lambda \end{vmatrix} = 0,$$

Here, one Eigen value of the matrix is attained as $-\delta$ and the other eigen values are roots of

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subsequent equation:

$$\lambda^2+(\xi_1+\xi_2+\beta+v_1+v_2+2\delta)\lambda+(\delta+v_1+v_2)(\xi_1+\xi_2+\delta)+\beta(\delta+v_1+\xi_1)=0$$

It is to be noticed that sum of the roots of the above quadratic equation is negative while product is positive. Hence roots of the quadratic equations must be negative.

Therefore, all the eigen values are negative and consequently, equilibria $E^{(1)}=(N_1(t),\,0,\,0)$ is locally asymptotically stable.

Theorem 4.2. The adopter free feasible equilibrium point $E^{(2)}=(N_1(t),N_2(t),\,0)$ is LAS if $\Re_0<1$.

Proof: The Jacobian matrix corresponding to the equilibria $E^{(2)}=(N_1(t),N_2(t),\,0)$ is:

$$J^{(2)}=\begin{bmatrix}-\xi_1-\delta & \xi_2 & v_1\\ \xi_1 & -\xi_2-\beta-\delta & -\alpha N_2+v_2\\ 0 & \beta & \alpha N_2-(\delta+v_1+v_2)\end{bmatrix}$$

The characteristic equation of $J^{(2)}$ is $|J^{(2)}-\lambda I|=0$ or

$$(-\delta-\lambda)\begin{vmatrix}1 & 0 & 0\\ \xi_1 & -\xi_1-\xi_2-\beta-\delta-\lambda & -\alpha N_2+v_2-\xi_1\\ 0 & \beta & \alpha N_2-(\delta+v_1+v_2)-\lambda\end{vmatrix}=0,$$

Therefore, one Eigen value of the matrix is attained as -δ and the remaining eigen values are roots of equation:

$$\lambda^2+(\xi_1+\xi_2+\beta+v_1+v_2+2\delta)\lambda+(\delta+v_1+v_2-\alpha N_2)(\xi_1+\xi_2+\delta)+\beta(\delta+v_1+\xi_1)=0$$

It is to be noticed that sum of the roots of the above quadratic equation is always negative so the both roots of this equation will be negative if product of roots is positive which is possible if $\delta + v_1 + v_2 - \alpha N_2 > 0$ which is possible if $\frac{\alpha N_2}{\delta + v_1 + v_2} < 1$.

Or $\Re_0 = \frac{\alpha N_2}{\delta + v_1 + v_2} = \frac{\alpha[(\Lambda_1 + \Lambda_2)\xi_1 + \Lambda_2 \delta]}{(\delta + v_1 + v_2)[\delta(\xi_2 + \beta + \delta) + \xi_1(\beta + \delta)]} < 1$.

Therefore, the adopter free feasible equilibrium point $E^{(2)} = (N_1(t), N_2(t), 0)$ is LAS if $\Re_0 <$ 1.

Theorem 4.3. The interior equilibrium point $E^* = (N_1^*(t), N_2^*(t), A^*(t))$ is LAS if $\alpha N_2^* < \min\{A^*, \delta + v_1 + v_2\}$.

Proof: The Jacobian to the equilibria $E^* = (N_1^*(t), N_2^*(t), A^*(t))$ is:

$$J^* = \begin{bmatrix} -\xi_1 - \delta & \xi_2 & v_1 \\ \xi_1 & -\xi_2 - \alpha A^* - \beta - \delta & -\alpha N_2^* + v_2 \\ 0 & \alpha A^* + \beta & \alpha N_2^* - (\delta + v_1 + v_2) \end{bmatrix}$$

The characteristic equation of J^* is $|J^* - \lambda I| = 0$ or

$$(-\delta - \lambda)\begin{vmatrix} 1 & 0 & 0 \\ \xi_1 & -\xi_1 - \xi_2 - \alpha A^* - \beta - \delta - \lambda & -\alpha N_2^* + v_2 - \xi_1 \\ 0 & \alpha A^* + \beta & \alpha N_2^* - (\delta + v_1 + v_2) - \lambda \end{vmatrix} = 0,$$

Here, one Eigen value of the matrix is attained as -δ and the other are roots of the equation: $\lambda^2 + C_1 \lambda + C_2 = 0$, where:

$$C_1 = \xi_1 + \xi_2 + \alpha(A^* - N_2^*) + \beta + v_1 + v_2 + 2\delta,$$

$$C_2 = (\delta + v_1 + v_2 - \alpha N_2^*)(\xi_1 + \xi_2 + \delta) + (\alpha A^* + \beta)(\delta + v_1 + \xi_1).$$

By Descartes's rule of signs, both roots of the above quadratic equation will be negative if C_1 is negative and C_2 must be positive and that is possible if $\alpha N_2^* < \min\{\alpha A^*, \delta + v_1 + v_2\}$.

4.3 Stability analysis of delayed model at interior equilibrium point E^*

For the delayed innovation diffusion model (1), the characteristic equation for J^* at E^* is obtained by taking the determinant

$$\left|J^* - \lambda I\right| = \begin{vmatrix} -\xi_1 - \delta - \lambda & \xi_2 & v_1 \\ \xi_1 & -\xi_2 - (\alpha A^* + \beta)e^{-(\delta+\rho+\lambda)\tau} - \delta - \lambda & -\alpha N_2^* e^{-(\delta+\rho+\lambda)\tau} + v_2 \\ 0 & (\alpha A^* + \beta)e^{-(\delta+\rho+\lambda)\tau} & \alpha N_2^* e^{-(\delta+\rho+\lambda)\tau} - (\delta + v_1 + v_2) - \lambda \end{vmatrix} = 0,$$

After solving this determinant, the following characteristic equation has been obtained:

$$\left(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3\right) + \left(B_1\lambda^2 + B_2\lambda + B_3\right)e^{-\lambda\tau} = 0, \quad (7)$$

Where:

$$A_1 = \xi_1 + \xi_2 + 3\delta + v_1 + v_2;$$
$$A_2 = (\xi_1 + \delta)(\delta + v_1 + v_2) + \xi_2(2\delta + v_1 + v_2) + \delta(\xi_1 + 2\delta + v_1 + v_2);$$
$$A_3 = \delta(\xi_1 + \xi_2 + \delta)(\delta + v_1 + v_2);$$
$$B_1 = [\alpha(A^* - N_2^*) + \beta]e^{-(\delta+\rho)\tau};$$
$$B_2 = \left\{[\alpha(A^* - N_2^*) + \beta](\xi_1 + \delta) + (\delta + v_1)(\alpha A^* + \beta) - \alpha N_2^*(\xi_2 + \delta)\right\}e^{-(\delta+\rho)\tau};$$
$$B_3 = \left\{[\delta(\alpha A^* + \beta) - \alpha N_2^*(\xi_2 + \delta)](\xi_1 + \delta) + \alpha N_2^* \xi_1 \xi_2 + \delta v_1(\alpha A^* + \beta)\right\}e^{-(\delta+\rho)\tau};$$

Model (1) will be locally asymptotically stable around E^* provided all the eigen values of the equation (7) has negative real parts. But being a transcendental equation, the investigation of the nature of all infinite eigen values i.e., the roots of the equation is quite hard. To avoid this challenging situation, we proceed to check the possibility where system losses its stability i.e., occurrence of Hopf bifurcation.

5. Hopf-bifurcation analysis

Here, we proceed to investigate the stability of model (1) for $\tau > 0$, and the Hopf bifurcation analysis of the system about $E^* = (N_1^*(t), N_2^*(t), A^*(t))$ and it will be endorsed in simulation section.

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Let τ be the bifurcation parameter to examine how the conditions of the stability of a positive steady state are affected by the parameter (time delay). To determine the instability brought on by a delay parameter in the system (1), let $\lambda = i\omega$, for any $\omega > 0$, be a root of (7) ($\tau > 0$). Substituting $\lambda = i\omega$ into Eq. (7), we get:

$$\left((i\omega)^3 + A_1(i\omega)^2 + A_2(i\omega) + A_3\right) + \left(B_1(i\omega)^2 + B_2(i\omega) + B_3\right)e^{-i\omega\tau} = 0,$$ Comparing the real and imaginary parts from the derived expression, we have:

$$B_2\omega\sin(\omega\tau) + (B_3 - B_1\omega^2)\cos(\omega\tau) = A_1\omega^2 - A_3, \tag{8}$$
$$B_2\omega\cos(\omega\tau) - (B_3 - B_1\omega^2)\sin(\omega\tau) = \omega^3 - A_2\omega \tag{9}$$

Eliminating trigonometry terms from the equations (8) and (9) by squaring and adding, we have: $\varphi(\omega^2) = \omega^6 + \chi_1\omega^4 + \chi_2\omega^2 + \chi_3 = 0,$ (10)

Or by taking, $\eta = \omega^2$, we have:

$$\varphi(\eta) = \eta^3 + \chi_1\eta^2 + \chi_2\eta + \chi_3 = 0, \tag{11}$$

where $\chi_1 = A_1^2 - 2A_2 - B_1^2$, $\chi_2 = A_2^2 - 2A_1A_3 + 2B_1B_3 - B_2^2$, $\chi_3 = A_3^2 - B_3^2$.

Under the condition of having at-least one positive root of the equation (11) indicates that the equation (10) will have purely imaginary roots. Therefore, the Hopf bifurcation will occur in the system (1) under the condition that the equation (11) has at-least one positive root.

The potential for a positive real root of (11) is explained by the following result:

Lemma 5.1 [25] If any of the following axioms are true, then (11) has at least one positive real root: (i) $\chi_3 < 0$, (ii) $\chi_3 \geq 0$, $\chi_1^2 - 3\chi_2 > 0$, and $\eta_0 > 0$ such that $\varphi(\eta_0) \leq 0$.

Thus, the model system (1) has purely complex eigen-values iff condition (i) or

(ii) in Lemma are confirmed. Also, the equation (11) can have at most three positive real roots, $\eta_i > 0;\ i = 1,2,3,$ it is also possible that there may exist three purely complex

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eigenvalues, $\lambda_i = i\omega_i = \pm i\sqrt{\eta_i}$, $i = 1, 2, 3.$

To explore the various values of τ_i corresponding to ω_i, solving the equation (10) and (11), we have threshold value of τ as:

$$\tau_i^{(j)} = \frac{1}{\omega_i}\cos^{-1}\left(\frac{(A_1\omega_i^2 - A_3)(B_3 - B_1\omega_i^2) + (\omega_i^3 - A_2\omega_i)B_2}{(B_3 - B_1\omega_i^2)^2 + (B_2\omega_i)^2}\right) + \frac{2j\pi}{\omega_i}, \quad j = 0, 1, 2, 3, ... \quad \text{for}$$

$i = 1, 2, 3$. (12)

The least value of $\tau_i^{(j)}$ at which the purely complex conjugate pair of eigenvalues of type $\lambda^* = \pm i\omega^*$ occur is therefore given as below:

$$\tau^* = \min_{\substack{i=1,2,3 \\ j=0,1,2,,,}} \{\tau_i^{(j)}\}, \quad \tau_i^{(j)} > 0.$$

Here, existence of pair of purely imaginary eigen values has already been explored, now to justify the transversality condition of Hopf bifurcation, we take τ and for $\tau > 0$, $\lambda = \sigma + i\omega$ is a value of Eq. (7), where $\omega>0$ is a real. By substituting $\lambda = \mu + i\omega$ into (7) we have:

$$\sigma^3 - 3\sigma\omega^2 + A_1(\sigma^2 - \omega^2) + A_2\sigma + A_3 + [\{B_1(\sigma^2 - \omega^2) + B_2\sigma + B_3\}\cos(\omega\tau) + (2B_1\sigma\omega + B_2\omega)\sin(\omega\tau)]e^{-\sigma\tau} = 0, \quad (13)$$

and

$$(-\omega^3 + 3\sigma^2\omega + 2A_1\sigma\omega + A_2\omega) + [-\{B_1(\sigma^2 - \omega^2) + B_2\sigma + B_3\}\sin(\omega\tau) + (2B_1\sigma\omega + B_2\omega)\cos(\omega\tau)]e^{-\sigma\tau} = 0, \quad (14)$$

Differentiating Eqns. (13) and (14) w.r.t. τ and using $\tau = \tau^*, \omega = \omega^*$, and $\sigma = 0$, the obtained expressions are as below:

$$K_1\left[\frac{d\sigma}{d\tau}\right]_{\tau=\tau^*} - K_2\left[\frac{d\omega}{d\tau}\right]_{\tau=\tau^*} = M_1, \quad (15)$$

$$K_2\left[\frac{d\sigma}{d\tau}\right]_{\tau=\tau^*} + K_1\left[\frac{d\omega}{d\tau}\right]_{\tau=\tau^*} = M_2. \quad (16)$$

Where:

$$K_1 = -3\omega^2 + A_2 + [\{B_2 + \tau(B_1\omega^2 - B_3)\}\cos(\omega\tau) + \omega(2B_1 - B_2\tau)\sin(\omega\tau)]e^{-\sigma\tau},$$

$$K_2 = 2A_1\omega - [\{B_2 + \tau(B_1\omega^2 - B_3)\}\sin(\omega\tau) + +\omega(2B_1 - B_2\tau)\cos(\omega\tau)]e^{-\sigma\tau},$$

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$$M_1 = [\omega(B_3 - B_1\omega^2)\sin(\omega\tau) - B_2\omega^2\cos(\omega\tau)]e^{-\sigma\tau},$$
$$M_2 = [\omega(B_3 - B_1\omega^2)\cos(\omega\tau) + B_2\omega^2\sin(\omega\tau)]e^{-\sigma\tau}.$$

Solving the equations (14) and (15) and get the value of $\frac{d\sigma}{d\tau}$ at $\tau = \tau^*, \omega = \omega^*$ and $\sigma = 0$ we have:

$$\left[\frac{d\sigma}{d\tau}\right]_{\tau=\tau^*} = \left[\frac{K_1M_1 + K_2M_2}{K_1^2 + K_2^2}\right]_{\tau=\tau^*} = \frac{\omega^{*2}}{K_1^2 + K_2^2}\left[\frac{d\varphi}{d\eta}\right]_{\omega=\omega^{*2}} \neq 0,$$

as
$$\begin{aligned} K_1M_1 + K_2M_2 &= \left(-3\omega^2 + A_2\right)M_1 + 2A_1M_2 - B_2^2\omega^2 + 2B_1\omega^2(B_3 - B_1\omega^2) \\ &= -\left(-3\omega^2 + A_2\right)\omega(\omega^3 - A_2\omega) + 2A_1\omega^2(A_1\omega^2 - A_3) - B_2^2\omega^2 + 2B_1\omega^2(B_3 - B_1\omega^2) \\ &= \omega^2[3\eta^2 + 2\chi_1\eta + \chi_2] \end{aligned}$$

Where $\varphi(\eta)$ is stated in Eq. (11). Hence, the roots (7) crossed over the vertical axis as the bifurcation parameter τ crosses over the threshold value. Hence, at the threshold value $\tau = \tau^*$, which is the smallest positive value of τ.

Theorem 5.2: [26] Suppose that $E^* = (N_1^*(t), N_2^*(t), A^*(t))$ exist and the condition in (H_1) are satisfied for the innovation diffusion model (1), then the conditions for $E^* = (N_1^*(t), N_2^*(t), A^*(t))$ to be LAS with τ are

(i) if $\tau \in [0, \tau^*)$, then E^* is LAS;

(ii) if $\tau \geq \tau^*$, the point E^* bifurcates into periodic orbits, i.e., it becomes unstable;

(iii) system (1) undergoes Hopf bifurcation at threshold value $\tau = \tau^*$ around E^* where

$$\tau^* = \frac{1}{\omega^*}\cos^{-1}\left(\frac{(A_1\omega^{*2} - A_3)(B_3 - B_1\omega^{*2}) + (\omega^{*3} - A_2\omega^*)B_2\omega^*}{(B_3 - B_1\omega^{*2})^2 + \left(B_2\omega^*\right)^2}\right).$$

6. **Numerical simulation**

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In this section, using MATLAB software numerical computational analysis has been carried out to justify results of the previous sections. For this, we have considered a fictitious set of parameters which has been demonstrated as below:

$$\frac{dN_1}{dt} = 0.25 - 0.1N_1(t) + 0.2N_2(t) - 0.2N_1(t) + 0.01A(t),$$

$$\frac{dN_2}{dt} = 0.4 + 0.1N_1(t) - 0.2N_2(t) - \left(0.4A(t-\tau) + 0.3\right)N_2(t-\tau) + 0.03A(t) - 0.2N_2(t)\,,$$

$$\frac{dA}{dt} = \left(0.4A(t-\tau) + 0.3\right)N_2(t-\tau) - (0.2 + 0.01 + 0.03)A(t)\,. \tag{17}$$

(i) When $\tau = 0$.

In the absence of the delay parameter, system shows its local asymptotic stability around the non- trivial equilibrium point E^*. By considering the different set of initial values, it has been verified from the figure 1 that system (17) converges to its stable equilibrium point E^*.

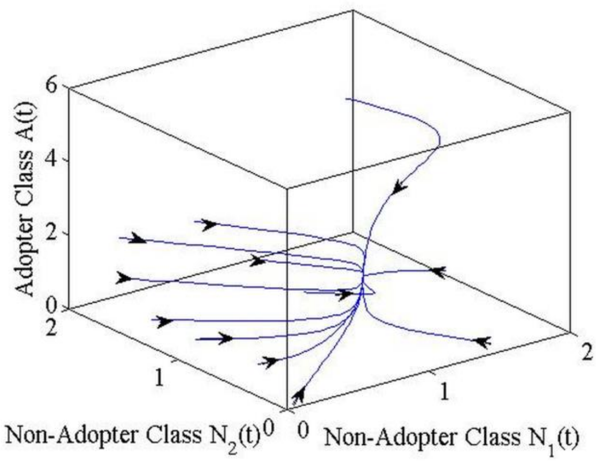

Fig. 1. All solution curves of the system (1) showing convergence towards stable equilibrium point.

(ii) When $\tau > 0$.

To analyse the importance of including the delay parameter in (1), we explore the system corresponding to the distinct values of the delay parameter τ corresponding to the initial

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values (0.3, 0.3, 0.1). For considering the value of τ = 3.9, the system shows convergence towards the stability around the non-trivial equilibrium point

Time series graph of all the three classes and phase space graph shown in Fig. 2 depicts the stable phenomena of the system for τ = 3.9.

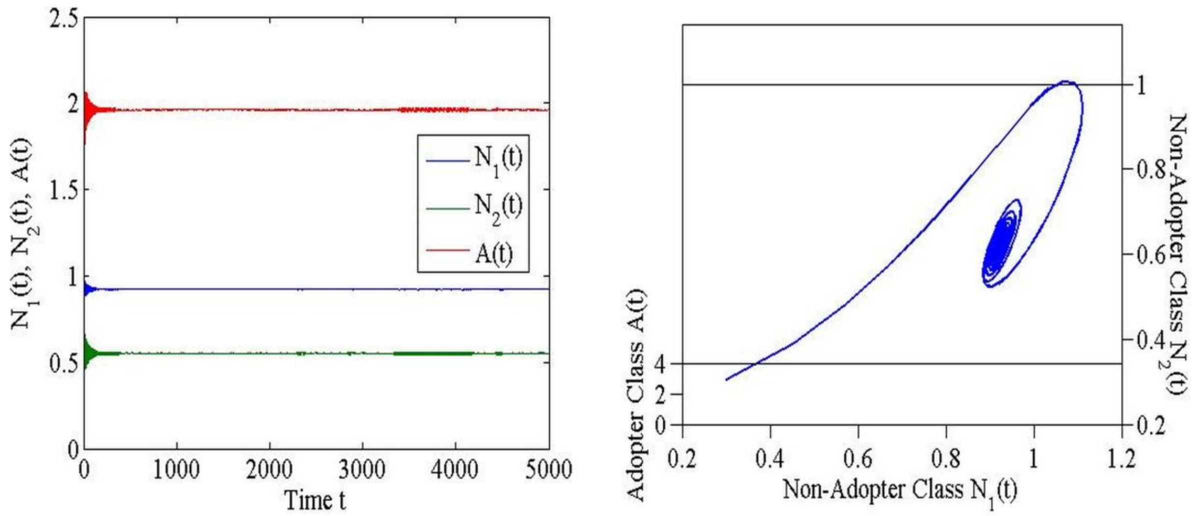

Fig. 2. Time series plots of all the Non adopter classes and the Adopter class and the phase space plot of the system for τ = 3.9 shows stability towards the interior point.

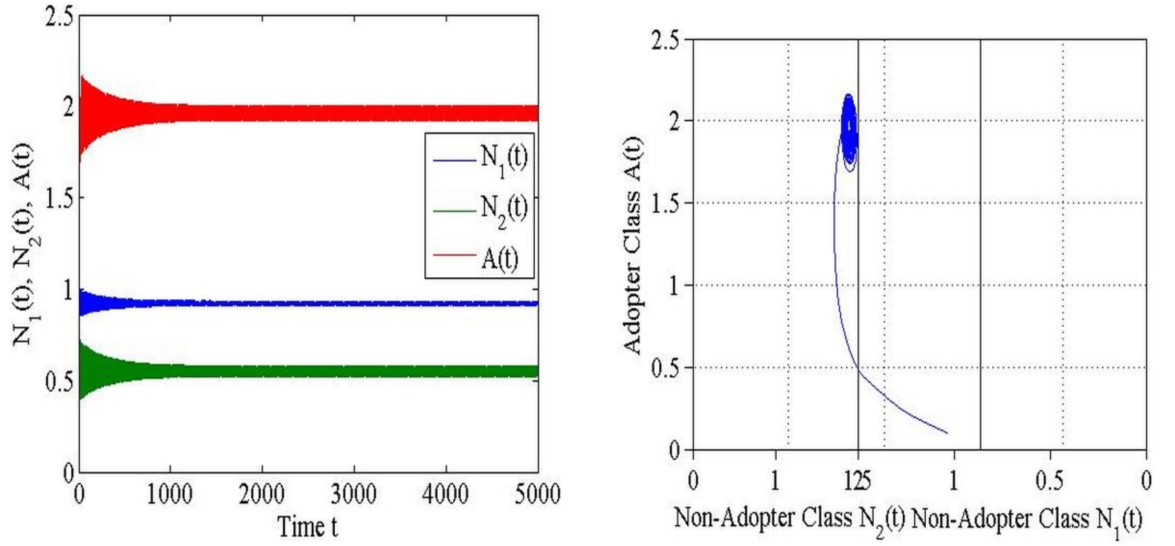

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Fig.3. Time series plots of all the Non adopter classes and the Adopter class and the phase space plot of the system for $\tau = 4.5 > 4.3 = \tau^*$ showing periodic oscillations towards the interior point.

With the continuous increase in delay, stability of system got disrupted. The critical value of delay period has been explored as $\tau = \tau^* = 4.3$. It has been investigated that before the critical value, system remain in stable position but after crossing its threshold value, Hopf bifurcation has been produced via limit cycles and the same phenomena has been depicted in Fig. 3 for τ = 4.5. The system (1) also begins to produce more stable limit cycles for a further value of the delay parameter τ = 4.8 and it has been through Fig. 4. The chaotic attractors encircling the inner equilibrium for τ = 15.8 are shown in Fig. 5, which also depicts the system's complex dynamics, as the existence of erratic oscillations. The time series plot and phase plane diagram in Fig. 3–Fig. 5 illustrate the complexity in the behaviour of model (1) with the occurrence of a Hopf bifurcation to irregular oscillations, and finally to chaotic circumstances.

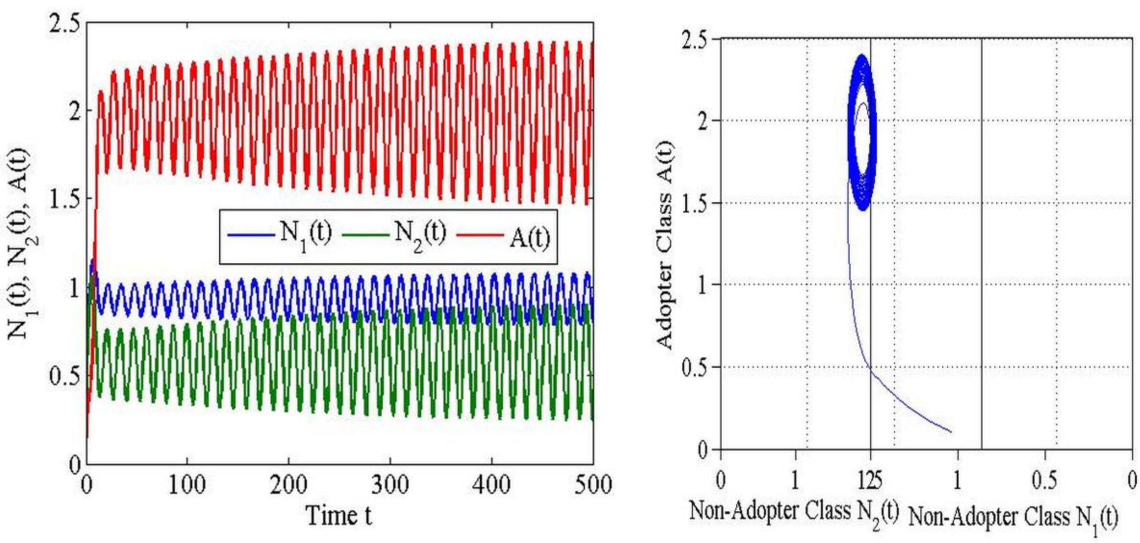

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Fig.4. Time series plots of all the Non adopter classes and the Adopter class and the phase space plot of the system for $\tau = 4.8 > 4.3 = \tau^*$ showing more stable limit cycle around the interior equilibrium.

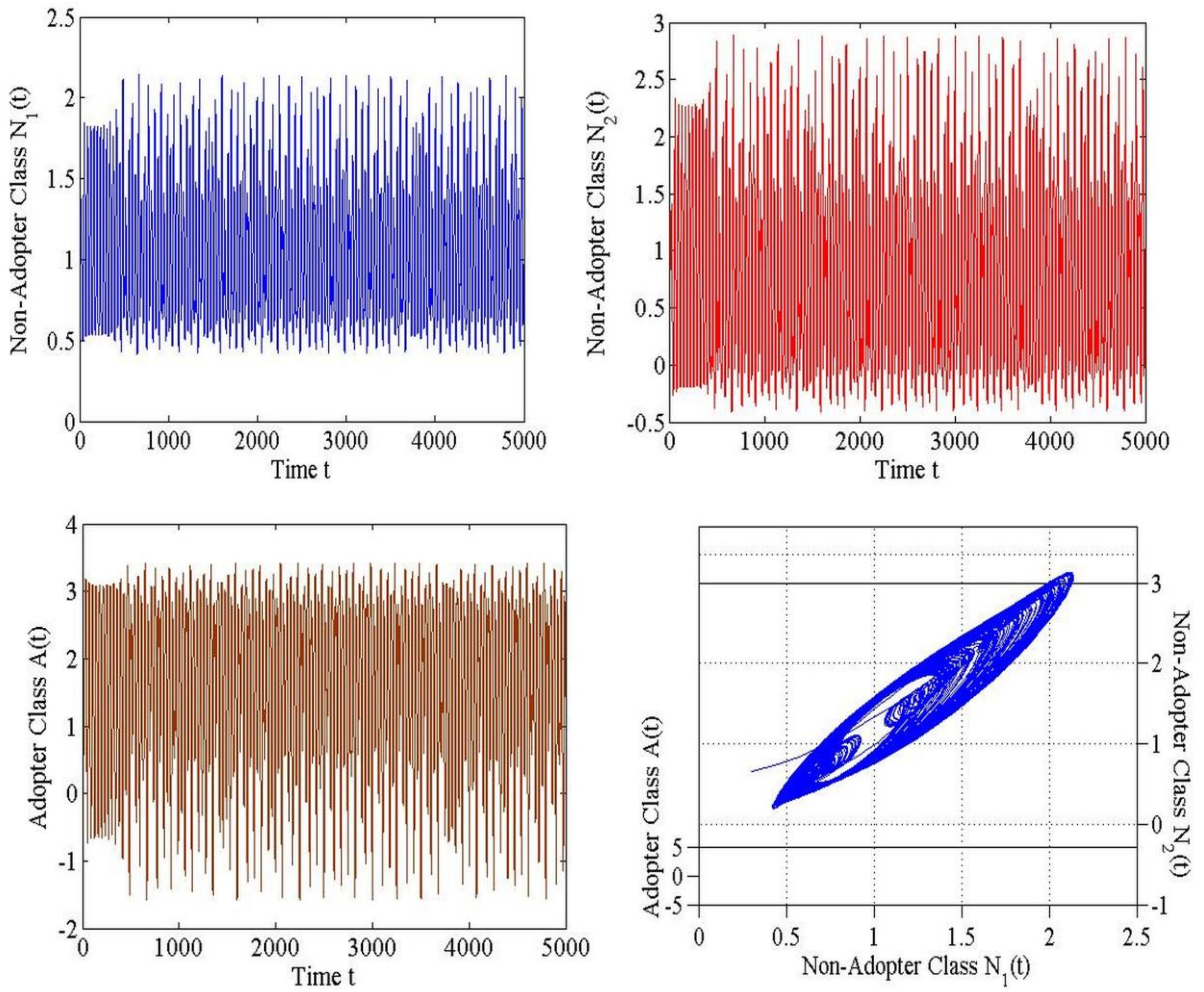

Fig.5. Description of complex dynamical behaviour of all the three classes corresponding to delay period $\tau = 15.8 > 4.3 = \tau^*$.

From the analytical as well as numerical simulations, it has been visualized that delay parameter played exemplary role in the dynamical behaviour of Non adopter and Adopter classes in (1). The findings demonstrated that the time delay parameter is essential for the innovation to succeed in different situations and that it has unstable effects on the model of

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innovation dissemination. It assisted the system (1) in producing irregular oscillations and ultimately chaos in the diffusion markets by converting it from a stable behaviour to Hopf bifurcation. Bifurcation diagrams corresponding to the delay parameter explain the significance of delay period on the dynamical behaviour of all the three classes in Fig. 6. producing complex situations in various markets for the innovation.

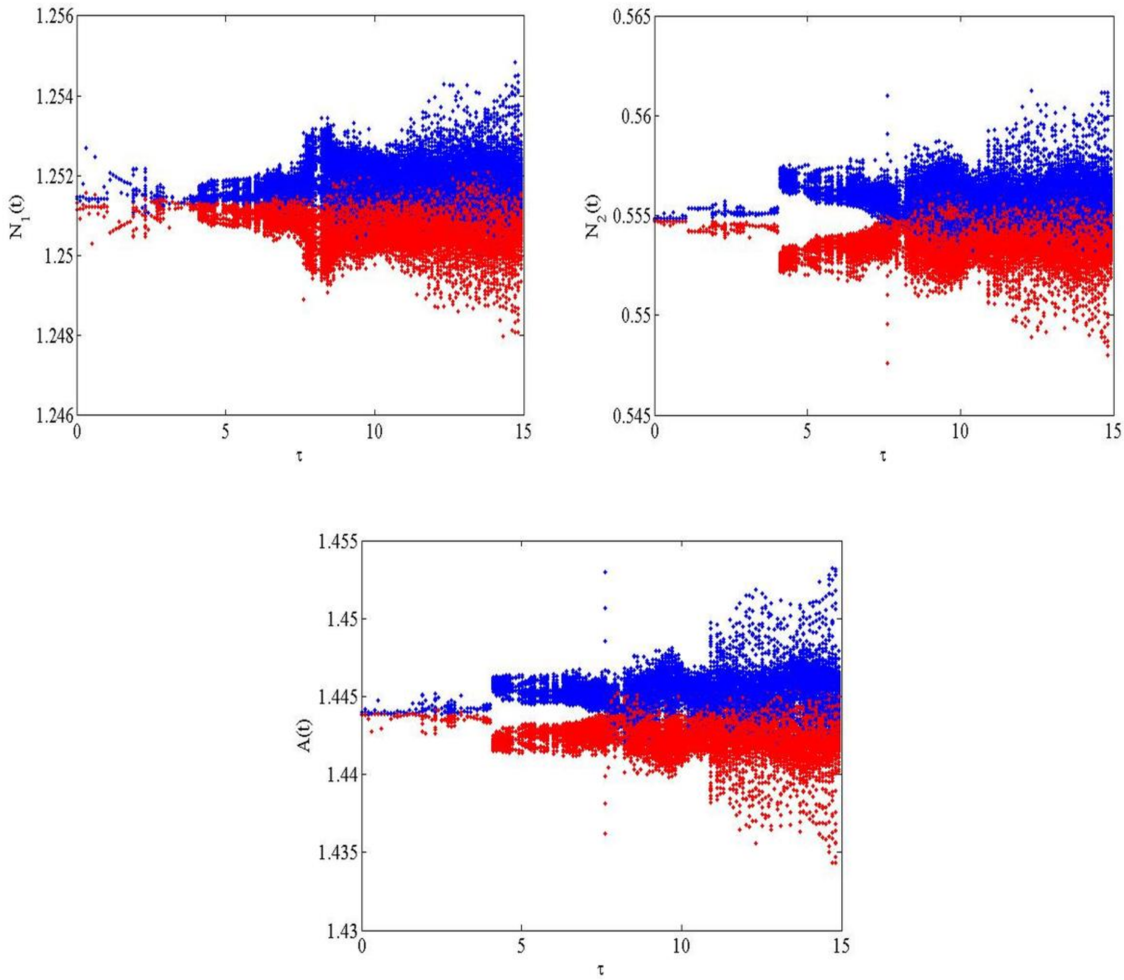

Fig.6. Bifurcation diagrams of all the three classes corresponding to the bifurcation parameter τ depicting the long-term effect of delay period on system (1).

7. Results and Discussion

Here, a mathematical system has been considered to examine the continuous diffusion model's dynamic behaviour when advertisements and interpersonal communication are

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included and delay period has been involved in adoption. The basic preliminary examinations of (1) supported the positivity of the solutions. Additionally, solutions of the proposed model lie within the confines of a defined region. Different possible steady points along with their stability has been investigated. Initially, stability has been explored in view of no delay period corresponding to Non adopter class-2 and adopter free class E_1, Adopter free equilibrium E_2 and lastly at non -trivial point E^*using the concept of Basic influence number $\Re_0 < 1$. It has been proved that adopter free steady state will be LAS if $\Re_0 < 1$ and if $\Re_0 > 1$ then system stability would get disturbed. In consideration of no delay period, system shows its stability around the non -trivial equilibrium point which has also been described through Fig. 1 in the numerical simulation section. It has been discovered that the delay period has the capability to move the system from stability to small periodic limit cycles and to chaotic situation via Hopf bifurcation. The critical value of delay period has been investigated as $\tau^* = 4.3$ i.e., when the delay period $\tau \in [0, \tau^*)$ then system stays in stable position but after crossing its threshold value $\tau^* = 4.3$ system moves from the stability to small periodic oscillation via limit cycles. More stable periodic oscillations have been shown in Fig. 4 for $\tau = 4.8$. Complex dynamic behaviour of the system generating chaotic attractors around E^* is explained in Fig. 5 at $\tau = 15.8$. The system's bifurcation diagram (6) has demonstrated how, with a little rise in the value of the parameter, the system transitions from stable dynamics to chaotic dynamics. Figure 6 shows that a slight increase in the number of advertisements causes the system to become more complex. Therefore, the crucial component of this work is the application of optimum advertising and interpersonal communication extents, both of which are essential for diffusion processes to succeed in a variety of markets.

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