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Classification and Properties of Atomic Classes in Lattices

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ABSTRACT

In this manuscript, we introduce a set of null sets, along with definitions for lattice measures of atoms and lattice semi-finite measures. Our key finding establishes that the lattice measure of any two atoms is either disjoint or identical. Additionally, we provide a proof demonstrating that the class encompassing all atoms within a lattice sigma algebra is countable. Ultimately, we affirm certain fundamental characteristics pertaining to atoms in a lattice sigma algebra.

Key words: Lattice, □□□□algebra, atom, measure

1. INTRODUCTION

In section 2, Tanaka[6] provides the definition of a lattice sigma algebra, while Anil Kumar et al.[1] expounds on the definitions of lattice measurable space, lattice measurable set, lattice measure space, and lattice σ–finite measure. This section also includes the proof of certain fundamental properties associated with lattice measurable sets.

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In the third section, we introduce a set of null sets, atoms, along with the definitions of lattice measures for atoms and lattice semi-finite measures. Within this context, we establish a significant result: the lattice measures of any two atoms are either disjoint or identical. Furthermore, we demonstrate that the class encompassing all atoms in a lattice sigma algebra is countable. Additionally, we present a theorem asserting that if a lattice sigma algebra is atom less, it must contain a countable number of disjoint non-empty lattice measurable sets. Lastly, we affirm certain elementary characteristics pertaining to atoms within a lattice sigma algebra.

2. Preparatory Measures:

In this section, we will provide a concise overview of established principles in lattice theory, drawing on well-known sources such as Birkhoff [2]. A structure $(L, \wedge, \vee)$ is designated as a lattice if it encompasses operations $\wedge$ and $\vee$ and adheres to the following conditions for any elements x, y, z within L:

(L1) Commutative law: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

(L2) Associative law: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$.

(L3) Absorption law: $x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$.

Hereafter, the lattice $(L, \wedge \vee)$ will often be written as L for simplicity. A lattice $(L, \wedge \vee)$ is called distributive if, for any x, y, z, in L.

(L4) Distributive law holds: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

A lattice L earns the designation of being complete if, for every subset A of L, it encompasses both the supremum $\vee A$ and the infimum $\wedge A$. In the case of a complete lattice, it inherently includes maximum and minimum elements, conventionally denoted as 1 and 0 or I and O, respectively [3].

A distributive lattice takes on the title of a Boolean lattice when, for any element x in L, there exists a singular and unique complement xc, satisfying the condition:

$x \vee xc = 1$ (L5) the law of excluded middle

$x \wedge xc = 0$ (L6) the law of non-contradiction

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Let L be a lattice and €: L → L be an operator. Then € is called a lattice complement in L if the following conditions are satisfied.

(L5) and (L6); $\forall$ x $\in$ L, x $\vee$ x^c = 1 and x $\wedge$ x^c = 0,

(L7) The law of contrapositive; $\forall$ x, y $\in$ L, x $\leq$ y implies $x^c \geq y^c$,

(L8) The law of double negation; $\forall$ x $\in$ L, $(x^c)^c$ = x.

In this document, we regard lattices as complete lattices that adhere to (L1) - (L8) with the exception of (L6), the law of non-contradiction.

Definition 2.1:

Unless specified otherwise, let X denote the entire set, and L be a lattice comprising subsets of X. A lattice L is referred to as a σ-Algebra if it satisfies the following conditions:

(1) $\forall$ h $\in$ L, $h^c \in$ L

(2) if $h_n \in$ L for n = 1, 2, 3 ....., then $\bigvee_{n=1}^{\infty} h_n \in$ L.

We denote □(L) = ß, as the lattice □-Algebra generated by L.

Example2.1. Let X = $\Re$, L = {measurable subsets of $\Re$} with usual ordering (≤). Here L is a lattice , □(L) = ß is a lattice □ - algebra generated by L.

Example2.2[3]. 1. {□□□X} is a lattice □-Algebra.

2. P(X) power set of X is a lattice □-Algebra.

Definition 2.2:

The ordered pair $(X$,B) is termed a lattice measurable space, where X is a set and B is a lattice, satisfying certain conditions that render it suitable for measurable space considerations.

Example2.3. X = $\Re$, □L = {All Lebesgue measurable sub sets of $\Re$ }

($\Re$,ß) is a lattice measurable space.

Definitition2.3. If μ: ß□□ R ∪ {□} satisfies the following properties, then m is called a lattice measure on the lattice □-Algebra □(L).

(1) μ(□) = μ(0) = 0.

(2) $\forall$ h, g $\in$□ ß, such that μ(h), μ(g) ≥ 0; h ≤ g □□μ(h) ≤ μ(g).

(3) $\forall$ h, g $\in$□ ß: μ(h $\vee$ g) + μ(h $\wedge$ g) = μ(h) + μ(g).

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(4) If h_n □□ ß, n ∈ N such that $h_1 \leq h_2 \leq ... \leq h_n \leq ....$, then μ(□ $\overset{\infty}{\underset{n=1}{\vee}}$ □h_n) = lim μ(h_n).

Let μ_1 and μ_2 be lattice measures defines on the same lattice □-Algebra ß. If one of them is finite, the set function μ(E) = μ_1(E) - μ_2(E), E ∈□ ß is well defined and countably additive on ß.

Example2.4[4]. Let X be any set. ß = P(X) be the class of all sub sets of X. Define for any A ∈ ß, μ(A) = +∞ if A is infinite

= |A| if A is finite. Where |A| is the number of elements in A. Then μ is a countable additive set function defined on ß and hence μ is a lattice measure on ß.

Definition 2.4: A set A is considered a lattice measurable set, or simply lattice measurable, if A belongs to the lattice B.

Example2.5. The interval (a, ∞) is a lattice measurable under usual ordering.

Example2.6. [0, 1] < $\Re$ is lattice measurable under usual ordering.

Definition 2.5: The lattice measurable space (X,B), combined with a lattice measure μ, is termed a lattice measure space, denoted by (X,B,μ).

Example2.7. $\Re$ is a set of real numbers, μ is the lattice Lebesgue measure on $\Re$ and ß is the family of all Lebesgue measurable subsets of real numbers. Then ($\Re$, ß, μ) is a lattice measure space.

Example2.8. $\Re$ be the set of real numbers and ß is the class of all Borel lattices, μ be a lattice Lebesgue measure on $\Re$ then ($\Re$, ß, μ) is a lattice measure space.

Definition 2.6: In the lattice measure space (X,B,μ), if the set X is finite, then the measure μ is referred to as a lattice finite measure.

Example2.9. The lattice Lebesgue measure on [0, 1] is a lattice finite measure.

Definition 2.7: If μ is a lattice finite measure, then the lattice measure space (X, B, μ) is termed a lattice finite measure space.

Example2.10. Let ß be the class of all Lebesgue measurable sets of [0, 1] and μ be a lattice Lebesgue measure on [0, 1] then ([0, 1], ß□□μ) is a lattice finite measure space.

Definition2.8. Let (X, ß□□μ) be a lattice measure space if there exists a sequence of lattices measurable sets { x_n } such that

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(i) $X = \overset{\infty}{\underset{n=1}{\vee}} x_n$ (ii) $\mu(x_n)$ is finite.

then μ is called a lattice σ – finite measure.

Example2.11. The lattice Lebesgue measure on $(\Re, \mu)$ is a lattice σ – finite measure since

$\Re = \overset{\infty}{\underset{n=1}{\vee}}$ (-n, n) and μ((-n,n)) = 2n is finite for every n.

Definition 2.9: If μ is a lattice σ-finite measure, then the lattice measure space $((X,B,\mu)$ is termed a lattice σ-finite measure space.

Example2.12. Let ß be the class of all Lebesgue measurable sets on $\Re = \overset{\infty}{\underset{n=1}{\vee}}$ (-n, n) and μ be a lattice Lebesgue measure on $\Re$ then ($\Re$, ß□□ μ) is a lattice σ – finite measure space

Theorem 2.1: Let $\{E_i\}$ be an infinite decreasing sequence of lattice measurable sets; that is, a sequence with $E_{i+1} < E_i$ for each $i \in N$. Let $\mu(E_i) < \infty$ for at least one $i \in N$. Then

$$\mu\left(\overset{\infty}{\underset{i=1}{\wedge}} E_i\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof: Let p be the least integer such that $\mu(E_p) < \infty$. Then $\mu(E_i) < \infty$, for all $I \geq p$.

Let $E = \overset{\infty}{\underset{i=1}{\wedge}} E_i$ and $F_i = E_i – E_{i+1}$.

Then the sets F_i's are lattice measurable and pair wise disjoint, clearly

$E_p – E = \overset{\infty}{\underset{i=p}{\vee}} F_i$. Therefore, $\mu(Ep - E) = \sum_{i=p}^{\infty} \mu(F_i) = \sum_{i=p}^{\infty} \mu(E_i - E_{i+1})$

But $\mu(Ep) = \mu(E) + \mu(E_p - E)$ and $\mu(E_i) = \mu(E_{i+1}) + \mu(E_i - E_{i+1})$

For all $i \geq p$ since $E < E_p$ and $E_{i+1} < E_i$, further, using the fact that $\mu(E_i) < \infty$, for all $i \geq p$, if follow that $\mu(Ep - E) = \mu(Ep) - \mu(E)$ and $\mu(E_i - E_{i+1}) = \mu(E_i) - \mu(E_{i+1})$ for all $i \geq p$.

Hence $\mu(Ep) - \mu(E) = \sum_{i=p}^{\infty} \mu(E_i) - \mu(E_{i+1})) = \lim_{n\to\infty} \sum_{i=p}^{n} (\mu(E_i) - \mu(E_{i+1})) = \lim_{n\to\infty} (\mu(E_p) - \mu(E_n))$

$= \mu(E_p) - \lim_{n\to\infty} \mu(E_n)$. Since $\mu(E_p) < \infty$, it gives $\mu(E) = \lim_{n\to\infty} \mu(E_n)$.

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Theorem 2.2: Let $\{E_i\}$ be an infinite increasing sequence of lattice measurable sets; that is, a sequence with $E_{i+1} > E_i$ for each $i \in N$. Let $\mu(E_i) < \infty$ for at least one $i \in N$. Then

$$\mu\left(\overset{\infty}{\underset{i=1}{\vee}} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof: If $\mu(E_p) = \infty$ for some $p \in N$, then the result is trivially true, since $\mu\left(\overset{\infty}{\underset{i=1}{\vee}} E_i\right) \geq \mu(E_p) = \infty$

For each $n \geq P$. Let $\mu(E_i) < \infty$. For each $i \in N$. Now $E = \overset{\infty}{\underset{i=1}{\vee}} E_i$, evidently $F_i = E_i - E_{i+1}$. Then the sets F_i's are lattice measurable and pair wise disjoint, clearly $E - E_i = \overset{\infty}{\underset{i=p}{\vee}} F_i$

$$\mu(E - E_i) = \mu(\overset{\infty}{\underset{i=1}{\vee}} F_i) = \sum_{i=1}^{\infty} \mu(F_i) = \sum_{i=1}^{\infty} \mu(E_{i+1} - E_i) = \mu(E) - \mu(E_i) = \sum_{i=1}^{\infty} (\mu(E_{i+1}) - \mu(E_i))$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\mu(E_{i+1}) - \mu(E_i)) = \lim_{n\to\infty} (\mu(E_{i+1}) - \mu(E_i))$$

it gives $\mu(E) = \lim_{n\to\infty} \mu(E_n)$.

§3. Describing the Class of Atoms in Lattice Sigma Algebras

Definition3.1: Let (Y, ß) be a lattice measurable space. A nonempty class N of sets, where N is contained in ß is called a class of null sets of ß

1) If $E \in N$ and $F \in ß$, then $E \wedge F \in N$, and

2) If $E_n \in N$, n=1, 2, 3...., then $\overset{\infty}{\underset{n=1}{\vee}} E_n \in N$.

Definition3.2: Let (Y, ß, μ) be a lattice measure space. A set E in ß is called a μ-atom if

1) $\mu(E) > 0$ and

2) If $F \in ß$ such that F is contained in E, then either $\mu(E\text{-}F) = 0$ or $\mu(F) = 0$.

Definition3.3: Let ß be a lattice σ – algebra on a set Y. A set E in ß is said to be an atom of ß if

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1) $E \neq \phi$ and
2) F in ß, F is contained in E implies $F = \phi$ or F = E.

Example 3.1[5]: The chain of natural numbers has just one atom, the number 2.

Example 3.2: The set of natural numbers under divisible order, all primes are atoms.

Note 3.1: A lattice σ – algebra ß of Y is said to be atomless if there are no atoms of ß.

Definition3.4: Lattice semi-finite measure: A lattice measure μ on a lattice σ – algebra ß of Y is said to be semi finite if $F \in$ ß, $\mu(F) = \infty$ implies there exists $E \in$ ß such that E is contained in F and $0 < \mu(E) < \infty$.

Result 3.1: Let (Y, ß, μ) be a lattice measure space, if E_1 and E_2 are atoms, then either $\mu(E_1 \Delta E_2) = 0$ or $\mu(E_1 \wedge E_2) = 0$ or (the lattice measure of any two atoms are either disjoint or identical)

Proof: Let $E_1$and E_2 are atoms. Since $E_1$is an atom by definition3.2, $E_2 \in$ ß such that $E_2$is contained in E_1 implies $\mu(E_1\text{-}E_2) = 0$ or $\mu(E_2) = 0$. Since E_2 is an atom $\mu(E_2) \neq 0$ implies $\mu(E_1\text{-}E_2) = 0$. By similar argument we have $\mu(E_2\text{-}E_1) = 0$. Now $E_1 \Delta E_2 = (E_1\text{-}E_2) \vee (E_2\text{-}E_1)$ implies $\mu(E_1 \Delta E_2) = \mu(E_1\text{-}E_2) + \mu(E_2\text{-}E_1)$ implies $\mu(E_1 \Delta E_2) = 0$. Also evidently $(E_1 \vee E_2) = (E_1 \wedge E_2) \vee (E_1 \Delta E_2)$ implies $\mu(E_1 \vee E_2) = \mu(E_1 \wedge E_2) + \mu(E_1 \Delta E_2)$ implies $\mu(E_1 \vee E_2) = \mu(E_1 \wedge E_2)$ (since $\mu(E_1 \Delta E_2) = 0$). Again if $\mu(E_1\text{-}E_2) \neq 0$ then $\mu(E_2) = 0$ now $E_1 \wedge E_2 \leq E_2$ implies $\mu(E_1 \wedge E_2) \leq \mu(E_2)$ implies $\mu(E_1 \wedge E_2) \leq 0$. But $\mu(E_1 \wedge E_2) \geq 0$ (by definition 2.3) therefore $\mu(E_1 \wedge E_2) = 0$. If $E_2\text{-}E_1 \neq 0$ similarly we get $\mu(E_1 \wedge E_2) = 0$.

Result 3.2: Let (Y, ß, μ) be a lattice measure space and μ is lattice σ – finite measure, then the class A of all atoms in a lattice σ-algebra ß is countable.

Proof: Let E_1, $E_2 \in A$ be any two sets by result 3.1. we have either $\mu(E_1 \Delta E_2) = 0$ or $\mu(E_1 \wedge E_2) = 0$.

If $\mu(E_1 \Delta E_2) = 0$ then the set $(E_1 \wedge E_2)$ represents an atom or if $\mu(E_1 \wedge E_2) = 0$ then $(E_1\text{-}E_2)$ and $(E_2\text{-}E_1)$ represents two disjoint atoms. Which implies two disjoint sets in ß – N. Continuing this process for E_1, E_2 ……, we get a countable collection of disjoint sets in ß – N which leads ß – N is countable.

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Theorem 3.1: Let μ be a lattice semi-finite measure on a lattice σ – algebra ß of X. Let N denotes the collection of sets of μ - measure zero. Then ß – N satisfies countable chain condition (ccc) if and only if μ is lattice σ – finite measure.

Proof: If μ is lattice σ – finite measure, it is obvious that ß – N satisfies ccc.

Conversely, if μ(X) < ∞, then there is nothing to prove.

If μ(Y) = ∞, choose E_1 in ß such that 0 < μ(E_1) < ∞. Choose E_2 in ß such that E_2 is contained in Y – E_1 and 0 < μ(E_2) < ∞. Continuing this process we get a sequence of disjoint sets E_1, E_2, …, in ß such that E_i in ß – N and μ(E_i) < ∞. If $\mu(Y - \bigvee_{i=1}^{\infty} E_i) < \infty$, then we have a decomposition of Y which implies that μ is σ – finite. $\mu(Y - \bigvee_{i=1}^{\infty} E_i) = \infty$, choose E_α in ß such that E_α is contained in $Y - \bigvee_{i=1}^{\infty} E_i$ and 0 < μ(E_α) < ∞, where α is the first countable ordinal. Proceeding inductively, since ß – N satisfies ccc, there exists a countable ordinal β such that $\mu(Y - \bigvee_{\alpha<\beta} A_\alpha) < \infty$. This implies that μ is lattice σ – finite measure.

Theorem 3.2: Let ß be a lattice σ – algebra of a set Y. ß is atomless if and only if every non empty set in ß contains countable number of disjoint non empty sets in ß.

Proof: Let E in ß be non empty set. Fix x ∈ E, we can choose E_1 in E such that $x \notin E_1$.

Now E_1 is non empty and E_1 is contained in E, choose E_2 in E such that $x \notin E_2$.

Now E_2 is non empty and E_2 is contained in E-E_1, choose E_3 in E such that $x \notin E_3$, continuing this process we get a family {E_α / α < β} of non empty disjoint sets contained in ß where β is the first uncountable ordinal.

The converse part is trivial.

Theorem 3.3: Let ß be a lattice σ – algebra of a set Y. Then is satisfies ccc if and only if ß is isomorphic to the power set, that is the class of all subsets, of some countable set.

Proof: We can prove this theorem by using theorem 3.1. and theorem 3.2. If ß satisfies ccc, the number of atoms of ß is countable. From Y remove all atoms of ß. In the view of above theorem3.2. the remaining part is empty. Hence it is isomorphic.

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Example 3.3: Take the numbers 0,1 and the fractions $\frac{m}{n}$, $0 < \frac{m}{n} < 1$ that is $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$,................ order as follows $0 < \frac{m}{n} < 1$ for all $\frac{m}{n}$; $\frac{m}{n} \leq \frac{r}{s}$ only if max(m, r) = r ; $\frac{m}{n}, \frac{r}{s}$ in comparable if $n \neq s$. clearly the fractions from 0 to 1 has a countable infinity of atoms and of dual of atoms.

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