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# ORIGINAL AND UNAMBIGUOUS STRUCTURES OF STABLE RAMANUJAN BIPARTITE GRAPHS

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**ABSTRACT:**

Marcus, A., Spielman, D.A., Srivastava, N [24] and Lubotzky, Phillips and Sarnak [22] presented the first explicit constructions of infinite families of Ramanujan graphs. These had degrees $p + 1$, for primes $p$. There have been a few other explicit constructions, presented by Friedman, J [18], all of which produce graphs of degree $q + 1$ for some prime power $q$. Gunnells, P [19] has proved that the existence of infinite families of bipartite Ramanujan of every degree. While today's proof of existence does not lend itself to an explicit construction, it is easier to understand than the presently known explicit constructions. In this article, author tries to present, interestingly express developments of an endless group of unequal Ramanujan bipartite graphs. Furthermore, author reconstruct to a portion of the known techniques for developing Ramanujan bipartite graphs and examine the computational work expected in really carrying out the different and recent development techniques.

**Keywords:** Ramanujan graphs, bipartite graphs, infinite families.

**2010 Mathematics Subject Classification:** Primary 05C50, Secondary 05C31

## 1.0 Introduction

Let $G = (V, E)$ be a on n vertices with vertex set $V$ and edge set $E$. A is bipartite if its vertex set can be partitioned into two nonempty subsets $X$ and $Y$ such that each edge of $G$ has one

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end in *X* and the other in *Y*, and a is *k*-partite if its vertex set can be partitioned into *k* nonempty subsets such that no edge in G has its both ends in the same subset. Degree of a vertex *v*, *d(v)*, is the number vertices adjacent to *v* and by *N(v)* we denote the neighbor set of *v*. The smallest and largest degrees of vertices in *G* are respectively denoted by $\delta(G)$ and $\Delta(G)$.

An essential topic in graph theory is the investigation of the spectral hole of a customary (undirected) graph, that is to say, the contrast between the two biggest eigenvalues of the adjacency matrix of such graph. Similarly, the indistinct hole of a bipartite, bi regular graph is the contrast between the two biggest particular upsides of its bi adjacency. Ramanujan graphs (separately Ramanujan bigraphs) are graphs with an ideal grisly hole. Unequivocal developments of such graphs have diverse applications in regions like software engineering, coding hypothesis, and packed detecting. Specifically, Burnwal, S.P. , M [8] proved that the first and third creators shows that the Ramanujan graphs can give the main deterministic answer for the purported network communication issues. Preceding the distribution of Marcus, A., Spielman, D.A. and Srivastava, N [23], the framework culmination issues had just a probabilistic arrangement. The unequivocal onstruction of Ramanujan graphs has been traditionally very much examined. A few unequivocal strategies are known, for instance, by crafted by Candès, E. and Recht, B [9], Cohen, M.B [11], Li [21], Marcus, A., Spielman, D.A. and Srivastava, N [27], Friedman J [17], Bibak et al. [6]. These techniques are drawn from ideas in direct polynomial mathematics, number hypothesis, portrayal hypothesis and the hypothesis of automorphic structures. Interestingly, nonetheless, no express techniques for building unequal Ramanujan bigraphs are known. There are two or three unique developments in Ballantine, C., Feigon, B., Ganapathy, R., Kool, J., Maurischat, and K.,Wooding, A [5], and Evra, S., Parzanchevski, O [14].

**Definition 1.1.** Alon N [1] A *d*-regular graph is said to be a Ramanujan graph if the second largest eigenvalue by magnitude of its adjacency matrix, call it $\lambda_2$, satisfies the following condition:

$$|\lambda_2| \leq 2\sqrt{d-1} \qquad (1)$$

A d-regular bipartite graph is said to be a bipartite Ramanujan graph if the second largest singular value of its biadjacency matrix, call it $\sigma_2$, satisfies

$$\sigma_2 \leq 2\sqrt{d-1} \qquad (2)$$

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Note the difference being made between the above two definitions.. If a graph is *d*-regular and bipartite, then it cannot be a Ramanujan graph, If $\lambda 2 = -d$, which contradicts (1). On the other hand, if it satisfies (2), then it is called a bipartite Ramanujan graph.

**Definition 1.2.** A *(dr , dc)*-biregular bipartite graph is said to be a *Ramanujan bigraph* if the second largest singular value of its biadjacency matrix, call it $\sigma 2$, satisfies

$$\sigma_2 \leq \sqrt{d_r - 1} + \sqrt{d_c - 1} \qquad (3)$$

It is easy to see that definition 1.2 contains the second case of definition 1.1 as a special case when $dr = dc = d$. A Ramanujan bigraph with $dr = dc$ is called an unbalanced Ramanujan bigraph stated by Babai. L [2].

The rationale behind the boundaries in the above definitions is given the following results.

In the interests of conciseness, the results are summarized and the bibliophile should access the unique sources for specific declarations.

**Remark 1.1** In the same way that a Ramanujan graph approximates the complete graph, a bipartite Ramanujan graph approximates a complete bipartite graph. We say that a *d*-regular graph is a bipartite Ramanujan graph if all of its adjacency matrix eigenvalues, other than *d* and *-d,* have absolute value at most $2\sqrt{d-1}$. The eigenvalue of *d* is a consequence of being *d*-regular and the eigenvalue of *-d* is a consequence of being bipartite. In particular, Baldoni,M.W., Ciliberto, C., and Cattaneo, G.M.P [3} recalled that the adjacency matrix eigenvalues of a bipartite graph are symmetric about the origin.

**Theorem 1.1** (Ballantine, C., Ciubotaru, D; see [4]) Fix *d* and let $n \to \infty$ in a *d*-regular graph with *n* vertices. Then

$$\lim_{n\to\infty} inf|\lambda_2| \geq 2\sqrt{d-1}. \qquad (4)$$

**Theorem 1.2** (Brito, G., Dumitriu, I., Harris, K.D see [7]) Fix $d_r$, $d_c$ and let $n_r$, $n_c$ approach infinity subject to $d_r = \frac{d_c n_c}{n_r}$

Then

$$\lim_{n\to\infty n_c\to\infty} \inf |\sigma_2| \geq \sqrt{d_r - 1} + \sqrt{d_c - 1} \qquad (5)$$

Considering that a d-standard graph has $d$ as its biggest eigenvalue $\lambda_1$, a Ramanujan graph is one for which the proportion $\frac{\lambda_2}{\lambda_1}$ is all around as little as could be expected, considering the Alon-Boppana bound of Theorem 1. 1. Essentially, given that a $(d_r\ ,\ d_c)-$ standard bipartite graph has $\sigma_1 = \sqrt{d_r d_c}$, a Ramanujan bigraph is one for which the proportion $\frac{\sigma_2}{\sigma_1}$ is all around as little as could be expected, considering Theorem 1.2.

From a specific perspective, Ramanujan graphs and Ramanujan bigraphs are inescapable. To be exact, on the off chance that $d$ is kept fixed and $n\to\infty$, the small part of $d$-normal, $n$-vertex graphs that fulfill the Ramanujan property approaches one; Chandrasekaran, K., and Velingker, A see [10]. Essentially, if $d_r\ ,\ d_c$ are kept fixed and $n_r\ ,\ n_c\to\infty$ (subject obviously to the condition that $d_r = \frac{d_c n_c}{n_r}$), then, at that point, the negligible part of $(d_r\ ,\ d_c)$- biregular graphs that are Ramanujan bigraphs approaches one; Davidoff, G., Sarnak and P., Valette, A see [7]. Notwithstanding, regardless of their pervasiveness, there are generally not many express strategies for building Ramanujan graphs.

## 2.0 Main Results

**Theorem 2.1.** Every $d$-regular graph $G$ has a signed adjacency matrix $S$ for which the minimum eigenvalue of $S$ is at least $2\sqrt{d-1}$.

**Proof.** If , $\mu_n \geq -2\sqrt{d-1}$then we know that $|\mu_i| \leq 2\sqrt{d-1}$ for all $1 < i < n$. Note that every 2-lift (The graphs $G^S$ that we form this way are called 2-lifts of *G*.) of a bipartite graph is also a bipartite graph.

One can use this theorem to build infinite families of bipartite Ramanujan graphs, because their eigenvalues are symmetric about the origin.

**Theorem 2.2.** If X is a k-bipartite, then $\lambda = k$ is an eigenvalue with multiplicity equivalent to the number of connected components of *X*.

**Proof.** Let $v = [x_1, x_2, \ldots . x_n]'$ be an eigenvector of $A$ with eigenvalue $k$. Without loss of over-simplification, let us assume that $|x_1| = \max_{1\leq i\leq n} |x_i|$.

Similarly let us assume$x_1 \geq k,\ k > 0$.

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Then

$$kx_1 = \sum_{j=1}^{n} a_{1j}x_j \leq \sum_{j=1}^{n} a_{1j}x_1 = kx_1$$

This implies, that each $j$ for which $a_{1j} \neq 0$, we should have $x_j = x_1$. In specific, this is valid for all $j$ for which $v_j$is contiguous $v_1$. Rehashing the contention with every one of the adjoining vertices, we derive that $x_j = x_1$ if $v_j$ is associated with $v_1$. As we might copy this contention for every part, the outcome is presently clear.

Subsequently, if $X$ is an associated $k$-ordinary bipartite graph, we might organize the eigenvalues as

$$k = \lambda_0(X) > \lambda_1(X) \geq \cdots \geq \lambda_{n-1}(X) \geq -k$$

It is easy to show that $-k$ is an eigenvalue of $X$ if and provided that $X$ is bipartite, in which case its variety is again equivalent to the quantity of associated parts. Any eigenvalue $\lambda_i \neq \pm k$ is alluded to as a nontrivial eigenvalue. We mean by $\lambda(x)$ the limit of the outright upsides of every one of the non-trifling eigenvalues. A Ramanujan multigraph is a k-bipartite graph fulfilling the constraint

$$\lambda(x) \leq 2\sqrt{k-1}.$$

A Ramanujan bipartite graph is a Ramanujan multigraph having no numerous edges or then again circles. The inspiration for these definitions will become obvious in earlier results obtained by several researchers. The meaning of such bipartite graphs will likewise be expounded upon later.

For the occasion, let us express that the unequivocal development of such bipartite graphs for a proper $k$ and $n \rightarrow \infty$ has just been depicted for the situation $k - 1$ is prime stated by Davidoff, G., Sarnak and P., Valette, A [12], Evra, S., Parzanchevski, O [14] or a great power (Deligne, P.: La conjecture de Weil. I see[13]) and it is as yet an open issue in the overall case.

Subsequently, the least difficult case that is open is when $k$=7. That is, we should develop a group of 7-customary graphs $x_i$ with $|x_i|$ tending to vastness whose relating nearness frameworks have non-trifling eigenvalues $\lambda$ fulfilling $|\lambda| \leq 2\sqrt{6}$. In this unique circumstance, Fan, J.L [15] develops what he calls 'nearly' Ramanujan bipartite graphs by utilizing the hypothesis of Hecke administrators. More unequivocally, he shows that for each $k$, there is a group of $k$-customary graphs $x_i$ with $|x_i|$ watching out for endlessness and the non-insignificant eigenvalues $\lambda$ of the relating nearness frameworks fulfill the disparity

$$\lambda(x) \leq d(k-1)\sqrt{k-1}.(x)$$

indicates the quantity of positive divisors of $k - 1$.

The complete bipartite graph as well as the bipartite graph $k_{rr}$ are effectively seen to be Ramanujan graph. The Petersen graph (see Figure 2.1) is a 3-normal graph whose contiguousness framework has trademark polynomial $(\lambda-1)^5(\lambda+2)^4(\lambda-3)$, and in this way is handily seen to be Ramanujan.

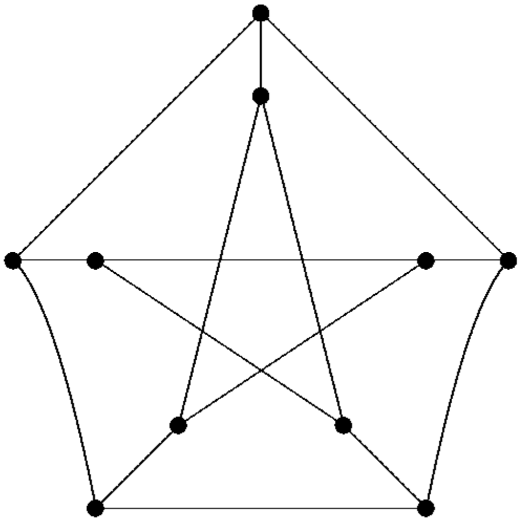

Figure 2.1 – The Petersen Graph

Feng, K., Li,W.-C.W [16] has exposed that random $k$-regular graphs are near to existence of Ramanujans bipartite graph in the sense that $\lambda_1$ (as defined above) satisfies the condition

$$\lambda_1 \leq 2\sqrt{k-1} + 2\log k + O(1).$$

**Remark 2.1.** It is substantial pointing out that proficient resolutions of the matrix completion problem do not really involve the existence of Ramanujan bipartite graphs of all sizes and degrees. It is adequate if the “gaps” in the permitted values for the degrees and the sizes are insignificant (Hall, C., Puder, D. and Sawin, W.F

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see [20]). If this extra sovereignty leads to significant simplification in the erection measures, then it would be a sensible tradeoff. However, research on this problem is still at a promising platform.

**Theorem 2.3.** For every $d \geq 5$, there exists an infinite sequence of *d*-regular bipartite Ramanujan graphs.

**Proof.** We know that every nontrivial eigenvalue of a complete *(c; d)*-biregular graph is zero. Thus the complete bipartite graph of degree *d* is Ramanujans graph.

By Lemma 3.1 of [4] and Let us assume tnat $G$ be a graph with adjacency matrix $A$ and universal cover $T$ . Then there exists a signing $s$ of $A$ so that all of the eigenvalues of $A_s$ are at most $\rho(T)$.

In particular, if $G$ is *d*-regular, there is a signing $s$ so that the eigenvalues of $A_s$ are at most $2\sqrt{d-1}$

Hence for every *d*-regular bipartite Ramanujan graph *G*, there is a *2*-lift in which every nontrivial eigenvalue is at most $2\sqrt{d-1}$.

As the 2-lift of a bipartite graph is bipartite and the eigenvalues of a bipartite graph are symmetric about 0, this *2*-lift is also a *d*-regular bipartite Ramanujan graph.

Thus for every *d*-regular bipartite Ramanujan graph *G*, there exists another *d*-regular bipartite Ramanujan graph with two times as many vertices.

## 3.0 Conclusion

As described in Sect. 2.0, a significant motivation for the study of explicit constructions of Ramanujan bipartite graphs comes from the matrix completion problem. This theme was explored in detail in Marcus, A., Spielman, D.A., Srivastava N [23] where it was shown that it is possible to guarantee exact completion of an unknown low-rank matrix, if the sampling set corresponds to the edge set of a Ramanujan bigraph. While that set of results is interesting in itself, it has New and explicit constructions of unbalanced Ramanujan left open the question of just *how* Ramanujan bigraphs are to be constructed. In the literature to date, there are relatively few explicit constructions of the Ramanujan graphs and

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balanced bigraphs, and no explicit constructions of an unbalanced Ramanujan bigraph. In this paper, we presented for the first time an infinite family of unbalanced Ramanujan bipartite graphs with explicitly constructed matrices with eigen values.

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