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Research Paper

Charatarization and estimation weighting probability distribution

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Abstract:

In this paper, we have introduced weighted Maxwell-Boltzmann distribution and abbreviated as WMD. Different characteristic properties of the introduced distribution have been studied in detail. Although the estimators are not derived in a closed form but parameters are estimated through the fitting of WMD to a particular data set using the technique of MLE. In order to show the validity, potentiality and flexibility of WMD in statistical modelling, we have fitted it to four different types of data sets. After the fitting of WMD to the considered data sets, comparison has been made between the special cases of WMD in terms of having least values of *BIC, AIC* & *AICC*. Random numbers from WMD are generated by using the Inverse *Cdf* method. Simulation has been carried out with the help of programming language R.

Keywords: Length-biased Maxwell-Boltzmann distribution (LBMD), Area-biased Maxwell-Boltzmann distribution (ABMD), Moment generating function, Characteristic function, Reliability, Entropy, Bonferroni curve, Lorenz curve, Order Statistics, *AIC, AICC, BIC.*

Introduction

The concept of weighted distributions can be traced from the work of Fisher in connection with his studies, on how methods of ascertainment can influence the form of distribution of recorded observations. Later it was developed and formulated in general terms

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by C.R. Rao in connection with modelling statistical data, where the usual practice of using standard distributions for the purpose was not found to be appropriate. It is quite obvious that while studying the real world random phenomena, the observations may be recorded with an amount of inherent bias. As a result of which these recorded observations will not have the original distribution unless every observation is given an equal chance of being recorded. C.R. Rao introduced a new class of distributions known as weighted distributions after analyzing the situations where observations are recorded with varying probabilities. The concept of weighted distribution is very important, because of the fact that weighted distributions take into consideration the method of ascertainment, by adjusting the probabilities of actual occurrence of events. We may arrive at wrong conclusions, while failing to make such adjustment. Thus, it is very imperative to use the concept of weighted distribution while dealing with a stochastic process in which the observations are being generated or recorded with varying probability. In order to increase the accuracy and to draw sound results, our main motive becomes to give importance to model specification. One of the unifying approaches for this purpose is to use the concept of weighted distributions. The importance of weighted distributions can be understood from L.L. Macdonald discussing the need for teaching weighted distribution theory. There are some traditional theories and practices which have been occupied with replication and randomization like environment Ric theory. Observations also fall in the non-experimental, non-replicated and non-randomized categories. Thus our main interest lies in drawing the inference about random phenomena with higher degree of accuracy. We can't guarantee the degree of accuracy of results unless the suitable and flexible model are used for statistical modelling. G. P. Patil and C. R. Rao quoted "Although the situations that involve weighted distributions seem to occur frequently in various fields, the underlying concept of weighted distributions as a major stochastic concept does not seem to have been widely recognized". Thus it is very essential to identify the stochastic processes where observations are recorded with varying probabilities so that the validity and importance of weighted distributions in statistical modelling can be understood. The concept of weighted distributions attracted a lot of researchers to contemplate on and to carry out research on the same. G .P. Patil and C.R. Rao studied weighted distributions and size biased sampling with applications to wildlife populations and human families. Van Deusen arrived at size biased distribution theory independently and applied it in fitting assumed distributions to data arising from horizontal point sampling. Subsequently, Lappi and Bailey used weighted distributions to analyze HPS diameter increment data. Rao studied the weighted distributions arising out of method of ascertainment. In fisheries, Taillie et. al modelled populations of fish stocks using weights. In ecology, Dennis and Patil used stochastic differential equations to arrive at weighted properties of size-biased Gamma distribution. K.G. Janardan characterized the weighted Lagrange distributions. K.G. Janardan and B.R. Rao studied the Lagrange distributions of the second kind and weighted distributions. G.S. Lingappaiah discussed Lagrange-negative binomial distribution in its simple and weighted forms. R.S. Ambagaspitiya defined weighted generalized Negative Binomial distribution. Asgharian, M. *et al.* worked on the length-biased sampling with right censoring. Gove studied the estimation and application of size-biased distributions in forestry. Kvam discussed about the Length bias in the measurements of Carbon Nanotubes. Dar *et al.* characterized the transmuted weighted Exponential distribution and discussed some of it's application. Reshi *et al.* worked on new moment method of estimation of parameters of size-biased classical gamma distribution.

Definition: Let us suppose that X be a continuous random variable of interest such that $X \sim f(x;\theta)$. However if the sample observations are selected with probability proportional to weighted function $w(x) = x\omega$, where $\omega > 0$ is the weight parameter. Then the distribution, with Pdf given by:

Derivation of Weighted Maxwell Distribution (WMD) In Physics and Chemistry there is a lot of applications of Maxwell (or Maxwell-Boltzmann) distribution. The Maxwell distribution forms the basis of the kinetic energy of gases, which explains many fundamental properties of gases, including pressure and diffusion. This distribution is sometimes referred to as the distribution of velocities, energy and magnitude of momenta of molecules. It was Tyagi and Bhattacharya who considered the Maxwell distribution as a lifetime model for the first time and discussed the Baye's and minimum variance unbiased estimation procedures for it's parameter and reliability function. Chaturvedi and Rani obtained classical and Baye's estimators for the Maxwell distribution, after generalizing it by introducing one more parameter. Empirical Baye's estimation for the Maxwell distribution was studied by Bekker and Roux Kazmi *et al.* carried out the Bayesian estimation for two component mixture of Maxwell distribution, assuming type I censored data. Herein, we considered Maxwell distribution and constructed its weighted version. The *Pdf* of a random variable X following Maxwell distribution with rate parameter θ is given by

$$f^{(x;\theta)} = \sqrt{2/\pi}\theta^{\frac{3}{2}}x^2\exp\left(-\theta x^2/2\right); x,\theta > 0$$

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Weight function: The weight function considered is $w(x) = x^{\omega}$, where $\omega > 0$ is the weight parameter. Therefore,

$$E_E w(x)] = \frac{2^{\omega/2+1}\Gamma((\omega+3)/2)}{\sqrt{\pi}\theta^{\omega}}$$

Now, from the definition(1), we will have the Pdf of WMD as given by (4):

$$f_{\omega}(x;\ \theta,\omega\Big) = \frac{\theta^{(\omega+3)/2}x^{\omega+2}\exp\left(-\theta x^2/2\right)}{2^{(\omega+1)/2}\Gamma((\omega+3)/2)}$$

Cdf, Reliability function and hazard rate of WMD are respectively given by (5), (6) and (7)

$$F\omega$$

$$R_{\omega}(x;\ \theta,\omega) = \frac{\Gamma((\omega+3)/2, \theta x^2/2)}{\Gamma((\omega+3)/2)}$$

$$h_{\omega}(x;\ \theta,\omega) = \frac{\theta^{(\omega+3)/2}x^{\omega+2}\exp\left(-\theta x^2/2\right)}{2^{(\omega+1)/2}\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{\omega+3}{2}, \theta x^2/2\right)}$$

Table-1: Special cases of WMD at different values of ω

weight (ω)	$\omega = 0$	$\omega = 1$	$\omega = 2$
Distribution	MD	LBMD	ABMD
$f_1(x;\theta)$	$\sqrt{\frac{2}{\pi}}\theta^{3/2}x^2e^{-\theta x^2/2}$	$(1/2)\theta^2x^3e^{-\theta x^2/2}$	$\{2^{-3/2}/\Gamma(5/2)\}\theta^{5/2}x^4e^{-\theta x^2/2}$
$F_{\omega}(x;\theta)$	$1-\frac{\Gamma(3/2,\theta x^2/2)}{\Gamma(3/2)}$	$1-\frac{\Gamma(2,\theta x^2/2)}{\Gamma(2)}$	$1-\frac{\Gamma(5/2,\theta x^2/2)}{\Gamma(5/2)}$

3 Structural properties of WMID

In this section, various structural properties of WMD has been discussed.

Theorem 3.l. The r^{th} moment about origin of a random variable X following WMD is given by:

$$\mu_r' = \left(\frac{2}{\theta}\right)^{r/2}\frac{\Gamma((\omega+r+3)/2)}{\Gamma((\omega+3)/2)};\ r = 1,2,3,\ldots$$

Proof.

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$$\mu_r' = \int_0^\infty x^r f_\omega(x,\theta,\omega)dx$$

$$\mu_r' = \int_0^\infty x^r \frac{\theta^{(\omega+3)/2} x^{\omega+2}}{2^{(\omega+1)/2}\Gamma((\omega+3)/2)} \exp(-\theta x^2/2)dx$$

$$\mu_r' = \frac{\theta^{(\omega+3)/2}}{2^{(\omega+1)/2}\Gamma((\omega+3)/2)} \int_0^\infty x^{\omega+r+2} \exp(-\theta x^2/2)dx$$

$$\mu_r' = \left(\frac{2}{\theta}\right)^{r/2} \frac{\Gamma((\omega+r+3)/2)}{\Gamma((\omega+3)/2)}$$

Moments: First four moments about origin are given as follows:

$$\mu_1' = \sqrt{\frac{2}{\theta}} \frac{\Gamma((\omega+4)/2)}{\Gamma((\omega+3)/2)}$$

$$\mu_2' = \left(\frac{2}{\theta}\right) \frac{\Gamma((\omega+5)/2)}{\Gamma((\omega+3)/2)}$$

$$\mu_3' = \left(\frac{2}{\theta}\right)^{3/2} \frac{\Gamma((\omega+6)/2)}{\Gamma((\omega+3)/2)}$$

$$\mu_4' = \left(\frac{2}{\theta}\right)^{2} \frac{\Gamma((\omega+7)/2)}{\Gamma((\omega+3)/2)}$$

Variance:

$$\sigma^2 = \frac{2[\Gamma((\omega+3)/2)\Gamma((\omega+5)/2) - \{\Gamma((\omega+4)/2)\}^2]}{\theta[\Gamma((\omega+3)/2)]^2}$$

Variation, Skewness and Kurtosis: Coefficient of variation, skewness and kurtosis are respectively given by (14), (15) and (16)

$$c.v. = \frac{\sqrt{\Gamma((\omega+3)/2)\Gamma((\omega+5)/2) - \{\Gamma((\omega+4)/2)\}^2}}{\Gamma((\omega+4)/2)}$$

$$\gamma_1 = \frac{\{\Gamma((\omega+3)/2)\}^2}{[\Gamma((\omega+3)/2)\Gamma((\omega+5)/2) - \{\Gamma((\omega+4)/2)\}^2]^{3/2}} \times \left[\Gamma\left(\frac{\omega+6}{2}\right) - 3\frac{\Gamma\left(\frac{\omega+4}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)} + 2\frac{\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^3}{\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^2}\right]$$

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$$\gamma_2 = \frac{\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^2}{\left[\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)-\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^2\right]^2}\times$$

$$\left[\ \Gamma\left(\frac{\omega+7}{2}\right)-\frac{4\Gamma\left(\frac{\omega+6}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)}+\frac{6\Gamma\left(\frac{\omega+5}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)}{\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^2}\right.$$

$$\left.-3\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^4/\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^3\right]$$

Theorem 3.2. The moment generating function and characteristic function of a random variable X following WMD are respectively given by (17) and (18).

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left(\frac{2}{\theta}\right)^{r/2}\frac{\Gamma((\omega+r+3)/2)}{\Gamma((\omega+3)/2)}$$

$$\Psi_x(t)=\sum_{r=0}^{\infty}\frac{(\iota t)^r}{r!}\left(\frac{2}{\theta}\right)^{r/2}\frac{\Gamma((\omega+r+3)/2)}{\Gamma((\omega+3)/2)}$$

Proof. From the definition of mgf we have:

Information measures of WMD

Theory which deals with the study of transmission, processing, utilization, and extraction of information is called Information theory. Abstractly, information can be viewed as the resolution of uncertainty. It was Claude E. Shannon who originally proposed the information theory in a landmark article. In this article, ”information” is thought of as a set of possible messages, where the goal is to send these messages over a noisy channel, and then to have the receiver reconstruct the message with low probability of error, in spite of the channel noise. The quantification, storage, and communication of information is having a key measure known as entropy. The amount of uncertainty in the value of a random variable or the outcome of a random process is measured in terms of entropy measure. A number of Information measures have been proposed by various authors. Shannon and Renyi entropy are two of them.

Table-2: Characteristics of WMD at diff. values of θ & ω

θ	ω	Mean	variance	c.v.	γ1	γ2	Renyi Entropy δ = 0.2	0.5	0.7	0.9999	Shannon Entropy
1	0	1.59577	0.45352	0.42202	0.48569	3.10816	1.41796	1.16050	1.07737	0.9961759	0.996154
	1	1.87997	0.46571	0.36299	0.40569	3.05929	1.45294	1.18975	1.10356	1.0192720	1.019250
	2	2.12769	0.47293	0.32321	0.35424	3.03698	1.47584	1.20662	1.11816	1.0318190	1.031796

2	0	1.12838	0.22676	0.42202	0.48569	3.10816	1.07139	0.81393	0.73080	0.6496023 0.649580
	1	1.32934	0.23285	0.36299	0.40569	3.05929	1.10637	0.84318	0.75699	0.6726989 0.672676
	2	1.50451	0.23646	0.32321	0.35424	3.03698	1.12927	0.86005	0.77159	0.6852455 0.685222
5	0	0.71365	0.09070	0.42202	0.48569	3.10816	0.61324	0.35578	0.27270	0.1914570 0.191435
	1	0.84075	0.09314	0.36299	0.40569	3.05929	0.64822	0.38503	0.29884	0.2145535 0.214531
	2	0.95153	0.09459	0.32321	0.35424	3.03698	0.67112	0.40190	0.31344	0.2271001 0.227077

From Table-2, it is quite evident that on increasing the value of weight parameter (w) for the fixed value of rate parameter (q), mean, variance and entropy increase whereas the coefficient of variation, skewness and kurtosis start decreasing. While on increasing the value of rate parameter for the fixed value of weight parameter, mean, variance and entropy decreases whereas the other three characteristics i.e. coefficient of variation, skewness and kurtosis remains unaffected due to their independence from q . It can also be seen from the last two columns of the table-2 that Renyi entropy approaches to Shannon entropy as the order (d) of Renyi entropy tends to 1.

Conclusion:

In this paper, various characteristic properties of WMD have been studied and discussed in detail. Three real life data sets and a simulated one is considered for illustrating the validity of WMD in statistical modelling. After the fitting of WMD to the considered data sets, different measures of goodness of fit like *AIC*, *BIC* and *AICC* have been computed for the special cases of WMD and are reported in table-3. The probability model with lowest *AIC*, *BIC* and *AICC* is considered to be the best fitted model. From table-3, it is evident that WMD possesses the least values of *AIC*, *BIC* and *AICC* followed by ABMD, followed by LBMD and then finally followed by

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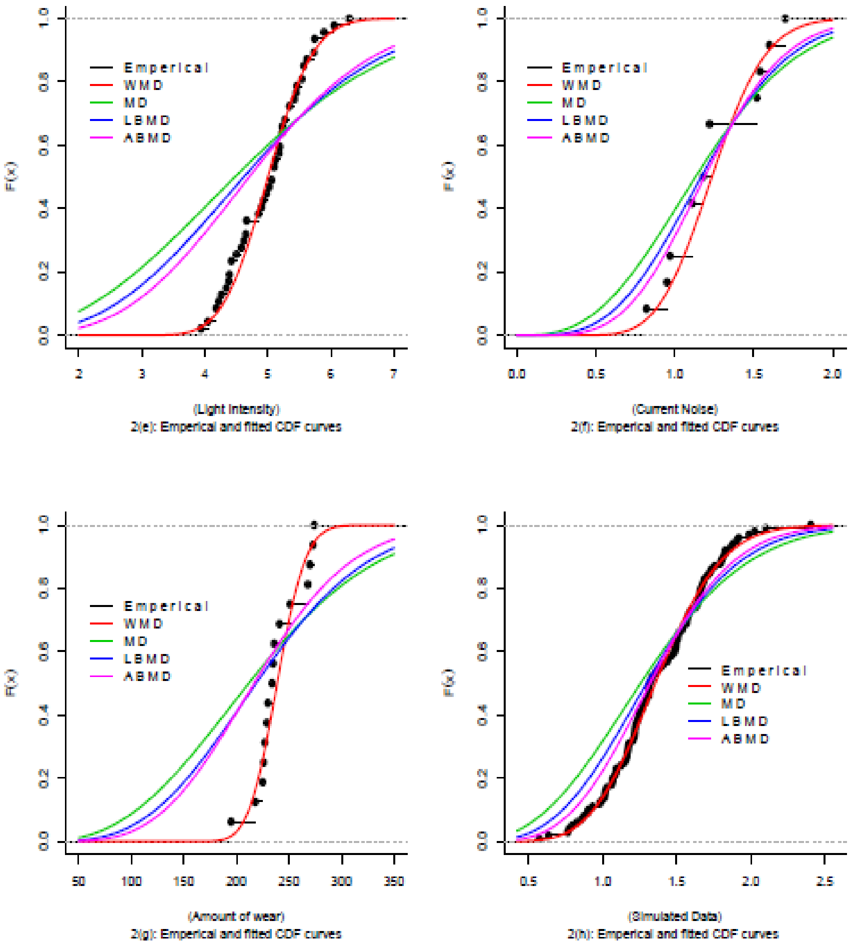

Fig. 2: Density and distribution curves fitted to four different types of data sets

Maxwell distribution (MD). Hence, it can be concluded that WMD proves to be more flexible and best fitted distribution in comparison to it's special cases in the current study. Therefore, distributions in the order of best fit for the considered data sets, are given as below: (*Best*) *WMD* −→ *ABMD* −→ *LBMD* −→ *MD* (*Good*) The main motive behind the construction of WMD and fitting of it's special cases to the considered data sets was to assess its potentiality and flexibility in modelling a particular data set. From the current study, it is concluded that if there is any intuition that the observations in a stochastic process are recorded with probabilities proportional to some weight function *w*(*x*,w), then it is better to contemplate on the need for studying weighted distributions and their application in modelling

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