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Existence of Bounded Solutions for Sylvester Matrix Dynamical Systems on Time Scales

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Abstract In this paper, the existence criteria for bounded solutions of Sylvester matrix dynamical systems on time scales are studied. The advantage of studying this system is that it unifies continuous and discrete systems. Sylvester matrix dynamical systems on time scales are considered, and for every Lebesgue-delta integrable function F on time scale T+, an investigation is carried out.

INTRODUCTION

Sylvester matrix and Lyapunov matrix differential equations find application in various areas of applied mathematics, including control systems, dynamic programming, optimal filters, quantum mechanics, and systems engineering. The objective of this paper is to provide the necessary and sufficient conditions for a linear Sylvester matrix dynamical system.

$X\Delta(t) = A(t)X(t) + X(t)B(t) + \mu(t)A(t)X(t)B(t) + F(t)$,

At least one Ψ-bounded solution for every Lebesgue Ψ-delta integrable function F on time scale T+ is aimed to be established in this paper, where Ψ is a regressive and rd-continuous matrix function [1]. The calculus of time scales, initiated by Stefan Hilger in 1988, was designed to unify discrete and continuous analysis. Dynamic equations on time scales have garnered significant recent attention in mathematics, shedding new light on the disparities between continuous differential equations and discrete difference equations. This approach also eliminates the need to prove a result separately for both types of equations. The primary objective, as highlighted in Bohner and Peterson's informative introductory text [2], is to demonstrate a result for a dynamic equation defined on a so-called time scale domain.

If T = Z, then μ(t) = 1 and the system (1.1) become Sylvester matrix delta difference system,

$\Delta X(t) = A(t)X(t) + X(t)B(t) + A(t)X(t)B(t) + F(t)$. (1.3)

In the above system (1.3), if we put A(t) = A1(t) In and B(t) = B1(t) In then the system becomes the following matrix difference system

$X(t + 1) = A1(t)X(t)B1(t)$.

The investigation at hand revolves around comprehending the behavior of Ψ-bounded solutions within the framework of the Sylvester matrix dynamical system (1.1). The primary objective of this inquiry is to establish a cohesive link, one that unifies the exploration across a spectrum of mathematical contexts, encompassing not only the components denoted as

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(1.2), (1.3), and (1.4), but also extending this unified analysis to matrix dynamical systems that operate on diverse time scales [3].

To set the stage, Fausset's comprehensive study [4] delved into the intricate facets of Sylvester matrix differential equations, meticulously dissecting analytical, numerical, and control-oriented aspects. On a parallel trajectory, Murthy [5] directed their attention towards the matrix difference system (1.3), rigorously investigating its existence and uniqueness [6].

In a more recent endeavor, Suresh Kumar and collaborators [7] embarked on an exploration that centered on the presence of Ψ-bounded solutions within the matrix difference system (1.4).

The underlying challenge that underpins this entire endeavor pertains to identifying the precise and comprehensive set of conditions that ensures the existence of a solution[8], one that not only satisfies the dynamics described by the system but also aligns with specified boundedness criteria. In this context, Coppel [9] stands as a seminal contributor, having presented classical findings that address this fundamental problem within both linear and nonlinear differential equations. Parallelly, Agarwal [10] ventured into a similar pursuit, concentrating on comparable results for difference equations.

Building upon this foundation, an array of scholars have delved into the intricate realm of Ψ-bounded solutions [11], particularly within the domain of linear ordinary differential equations and difference equations. Their collective work, as evidenced by citations such as, illustrates the sustained interest in this multifaceted problem[12].

Stepping into more recent territory, the concept of Ψ-bounded solutions has found a broader scope of application through its extension to the realm of Lyapunov matrix differential and difference equations. This evolution is underscored by scholarly endeavors such as [13]which serve to expand the horizons of knowledge in this domain.

At the crux of this current study lies an ambitious goal: to synthesize and consolidate the diverse outcomes and insights gleaned from the study of Ψ-bounded solutions within linear differential equations [14], as well as within linear difference equations [15]. This amalgamation of findings extends seamlessly into the realm of Sylvester matrix dynamical systems on time scales. The focal point within this endeavor is the establishment of a definitive condition, both necessary and sufficient, that dictates the existence of a minimum of one Ψ-bounded solution for the linear non-homogeneous Sylvester matrix dynamical system (1.1), where the temporal framework is defined by the time scale T+. The methodological instrument employed in this pursuit is the innovative technique of Kronecker product of matrices. It is important to emphasize that this condition applies universally to each Lebesgue Ψ-delta integrable function F that operates within the same time scale.

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1. Preliminaries

In this section, we present some basic notations, definitions and results of time scales and Kronecker product of matrices. For more details about time scales refer [5] and Kronecker products of matrices refer [3].

Definition 2.1 as given in [5] outlines the forward jump operator σ : T → T, the backward jump operator ρ : T → T, and the graininess function μ : T → R+, which are formulated as follows:

The forward jump operator σ(t) is defined as the infimum of s in T such that s is greater than t.

The backward jump operator ρ(t) is defined as the supremum of s in T such that s is less than t.

The graininess function μ(t) is determined as the difference between σ(t) and t for each t in T. Additionally, if the condition σ(t) = t holds, the corresponding t is termed "right-dense"; otherwise, if ρ(t) = t, it is referred to as "right-scattered." Similarly, if ρ(t) = t, the t is deemed "left-dense," and otherwise, it is labeled "left-scattered."

Definition 2.2, as presented in [5], introduces the concept of Tk, where its value depends on the presence of a left-scattered maximum m within the time scale T. If T encompasses a left-scattered maximum m, then Tk is defined as the set T excluding the element m. In the absence of a left-scattered maximum, Tk remains equivalent to T.

Given the assumption that T is unbounded both above and below, it follows that Tk is consistently equivalent to T throughout the entirety of this paper.

2. Sylvester Matrix Dynamical Systems on Time Scales

In this section, a necessary and sufficient condition is derived for the existence of a Ψ-bounded solution for the Sylvester matrix dynamical system (1.1), utilizing a Ψ-deltaintegrable matrix function F on the time scale T+. Additionally, a result pertaining to the asymptotic behavior of the Ψ-bounded solution of (1.1) is established.

Theorem 3.1 asserts that if A(t) and B(t) represent regressive and rd-continuous d x d matrices defined on T+, and the system (1.1) possesses at least one Ψ-bounded solution on T+ for each Lebesgue Ψ-deltaintegrable matrix function F : T+ → Rd×d, then a crucial condition must hold true: there exists a positive constant N such that the following inequalities are satisfied:

| (Z(t) ⊗ Ψ(t) Y(t)) Q1 (Z^(-1)(σ(s)) ⊗ Y^(-1)(σ(s)) Ψ^(-1)(s)) | ≤ N for v ≤ σ(s) ≤ t | (Z(t) ⊗ Ψ(t) Y(t)) Q2 (Z^(-1)(σ(s)) ⊗ Y^(-1)(σ(s)) Ψ^(-1)(s)) | ≤ N for v ≤ t ≤ s (3.1)

The proof of this theorem commences by assuming that the equation (1.1) possesses at least one Ψ-bounded solution on T+ for each Lebesgue Ψ-deltaintegrable matrix function F : T+ → Rd×d. Through a sequence of logical deductions, it is established that the fundamental matrix

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W(t) of the related system (2.2) satisfies the prescribed condition (3.1), thus validating the theorem.

Conversely, if the condition (3.1) holds true for a given positive constant N, the theorem establishes that the Sylvester matrix dynamical system (1.1) will invariably harbor at least one Ψ-bounded solution on T+ for every Lebesgue Ψ-deltaintegrable matrix function F on the same time scale.

The subsequent theorem addresses sufficient conditions for characterizing the asymptotic behavior of Ψ-bounded solutions within the dynamical system (1.1).

Remark 3.3. *Theorem 3.2 is no longer true if we require that the function F be Ψ-bounded on $\mathbb{T}^+$, instead of the condition (2) and it doesnot apply even if the function F is such that* $\lim_{t\to\infty} |\Psi(t)F(t)| = 0$.

The following example illustrates Remark 3.3.

Example 3.4. Consider (1.1) with $A(t) = B(t) = O_2$ and

$$F(t) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ 0 & \sqrt{(t+1)^3} \end{bmatrix}.$$

Then, $Y(t) = Z(t) = I_2$ are the fundamental matrices for (2.3) and (2.4) respectively. Consider

$$\Psi(t) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{(1+t)^2} \end{bmatrix},$$

then there exist projections

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

satisfies the condition (3.1) with $N = 1$. In addition, the hypothesis (1a) of Theorem 3.2 is satisfied. Because

$$\Psi(t)F(t) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ 0 & \frac{1}{\sqrt{t+1}} \end{bmatrix},$$

the matrix function F is not Lebesgue Ψ-deltaintegrable on $\mathbb{T}^+$, but it is Ψ-bounded on $\mathbb{T}^+$, with $\lim_{t\to\infty} |\Psi(t)F(t)| = 0$. The solutions of the system (1.1) are

$$X(t) = \begin{cases} \begin{bmatrix} \ln(t+1) + c_1 & c_2 \\ c_3 & \frac{2}{5}\sqrt{(t+1)^5} + c_4 \end{bmatrix} & \text{when } \mathbb{T} = \mathbb{R} \\ \begin{bmatrix} \sum_{k=1}^{t} \frac{1}{k} + c_1 & c_2 \\ c_3 & \sum_{k=1}^{t} \sqrt{k^3} + c_4 \end{bmatrix} & \text{when } \mathbb{T} = \mathbb{Z}. \end{cases}$$

It is easily seen that $\lim_{t\to\infty} \|\Psi(t)X(t)\| = +\infty$, for all $c_1, c_2, c_3, c_4 \in \mathbb{R}$. It follows that the solutions of the system (1.1) are Ψ-unbounded on $\mathbb{T}^+$.

It is observed that the asymptotic properties of the solution components are not uniform. Specifically, the element in the second row and second column exhibits unbounded behavior on T+, while the remaining elements remain bounded. However, it is noteworthy that all solutions of (1.1) are bounded on T+ and exhibit a limit such that lim(t)X(t) = 0. This

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observation underscores the disparate asymptotic characteristics displayed by the individual solution components. The uniformity in the asymptotic properties of the solution components is demonstrated through the utilization of a matrix function Ψ, as opposed to a scalar function. This choice allows for the consistent characterization of the solution components' asymptotic behavior.

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