ISSN PRINT 2319 1775 Online 2320 7876

 UGC CARE Listed (Group -I) Journal Volume 11, Iss 12, 2022

SPECTRAL PROPERTIES OF ZERO DIVISOR GRAPH OF A FINITE LATTICE

S. Nithya[1], S. Valarmathi[2] and G. Manju[3]

[1]Assistant Professor, PG and Research Department of Mathematics,
St. Xavier's College(Autonomous), Palayamkottai.
Email:nithyasxc@gmail.com

[2]Research scholar (Reg. No. 19221282092011), PG and Research Department of Mathematics, St. Xavier's College (Autonomous),
Manonmaniam Sundaranar University, Tirunelveli, Tamilnadu, India.
Email: valarmathirenisha@gmail.com

[3]Research scholar (Reg. No. 23111282092008), PG and Research Department of' Mathematics, St. Xavier s College (Autonomous),
Manonmaniam Sundaranar University, Tirunelveli, Tamilnadu, India.
Email: manjuganesan20@gmail.com

Abstract. The goal of this work is to provide some fresh insights on the associated adjacency matrix of the zero divisor graph of $(D_n, |)$, where n is a positive integer and $D_n$is the set of all divisors of n, and its finite product. In this paper we examine eigenvalues, determinant, nullity, energy, resolvant energy, estrada index, resolvant estrada index and k-th spectral moment to the corresponding their adjacency matrix using scilab software.

Keywords: Adjacency Matrix, Zero divisor graph of a Lattice, Eigenvalues, Energy, Resolvant energy, Estrada index, Spectral moment.

1. Introduction

A graph is an abstract representation of a collection of items in which some pairs of objects are linked together by links; in this case, the linkages have a zero divisor. The items in the network are represented by mathematical vertices, or abstractions, and the connections that connect certain pairs of vertices known as edges.

Let $L = (D_n, |)$, be a lattice with the least element 1. We associate a simple graph $\Gamma(L)$ to L with the vertex set $Z(L)^* = Z(L)\setminus\{1\} = \{x | x \wedge y = gcd(x,y) = 1$ for some $y \neq x \neq 1\}$, the set of non-zero zero divisors of L and distinct $x, y \in Z(L)^*$ are adjacent if and only $if\ x \wedge y = gcd(x,y) = 1$. There are many papers which interlink graph theory and lattice theory [4, 10, 19, 29, 33]. These papers discuss the properties of graphs derived from partially ordered sets and lattices. In [19], E. Estaji and K. Khashyarmanesh associated to any finite lattice L, a simple graph $G(L)$ whose vertex set is $Z(L)^*$ and two vertices x and y are adjacent if and only if $x \wedge y = 0$. They studied the structure of $G(L)$ and connections between graph theoretical properties and lattice theoretical properties. In this paper we construct zero divisor graphs for cartesian

ISSN PRINT 2319 1775 Online 2320 7876

 UGC CARE Listed (Group -I) Journal Volume 11, Iss 12, 2022

product of finite lattices and then we obtain adjacency matrices corresponding to these graph after that we determine some results of the adjacency matrices. Throughout the paper $\Delta(\Gamma(L))$ denotes the maximum degree of graph.

The following finite lattices are analyse in order to their zero divisor graph of adjacency matrices: a) $D_{p^n} \times D_{p^n}$, b)$Dp^k q$, c) D_{pqr}, d) D_{pqrs}, e) D_{pqrst}, f)$D_{pq^2r^3}$, g) $D_{pq^3r^4}$, h) $D_{(pqr)^2}$, where p,q,r,s and t are distinct prime numbers.

Basic reference for graph theory [5] for lattice theory [11] and for matrix theory [3]. The adjacency matrix corresponding to the graph G is defined as $A = [a_{ij}]$, where

$$a_{ij} = \{1, \quad v_i \text{ and } v_j \text{ are adjacent}, v_i, \quad v_j \in V(G) \quad 0, \quad \text{otherwise}$$

Since A is a real and symmetric matrix of order n, its eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real numbers. These eigenvalues form the spectrum of G [6, 7]. The k-th spectral moment of the graph G is defined as

(1) $M_k = M_k(G) = \sum_{i=1}^{n} \lambda_i^k$

As well known [6, 7], $M_0 = n$, $M_1 = 0$, $M_2 = 2m$, and $M_k = 0$ for all odd values of k if and only if G is bipartite.

The energy of the graph G is defined as the sum of absolute values of its eigenvalues, i.e,

(2) $E = E(G) = \sum_{i=1}^{n} |\lambda_i|$

This spectrum based graph invariant was introduced in the 1970's [20]. Since then, its theory has been extensively elaborated [30] resulting in several hundreds of published papers [22]. Also a number of other "graph energies" have been introduced, based on matrices other than $A(G)$ [21, 27, 31, 32] The resolvent energy of a graph G is defined as

(3) $ER = ER(G) = \sum_{i=1}^{n} \frac{1}{n-\lambda_i}$

Note that the term $\frac{1}{n-\lambda_i}$ is always positive valued. The resolvent energy, introduced in[25] (see also [26]), is similar to, but not identical with, an earlier studied spectrum based graph invariant, put forward by Estrada and Higham [28] and defined as

(4) $EE_r = EE_r(G) = \sum_{i=1}^{n} \frac{M_k(G)}{(n-1)^k}$

Its relation to the resolvent of the adjacency matrix was recognized by Benzi and Boito [2], by showing that

(5) $EE_r = \sum_{i=1}^{n} \frac{n-1}{n-1-\lambda_i}$

in view of which EE_r has been named "resolvent Estrada index". Note that according to Eq.(5) , EE_r is undefined (i.e., infinite) in the case of the complete graph K_n. Additional properties of EE_r can be found in the recent papers [8, 9, 23, 24]. Estrada et al. [18] proposed to scope this bit of

ISSN PRINT 2319 1775 Online 2320 7876

Research paper © 2012 IJFANS. All Rights Reserved, UGC CARE Listed (Group -I) Journal Volume 11, Iss 12, 2022

information by using a Gaussian matrix function, which gives rise to the Gaussian Estrada index, $H(G)$. $H(G)$ can be defined as

(6) $H = H(G) = trace(e^{-A^2}) = \sum_{i=1}^{n} e^{-\lambda_i^2}$

Gaussian Estrada index H is able to describe the partition function of quantum mechanics systems with Hamiltonian A^2[35]. It gives more weight to eigenvalues close to zero and ideally complements the Estrada index. Moreover, it is also related to the time-dependent Schr$\ddot{o}$inger equation with the squared Hamiltonian. Based on numerical simulations, H is found to be effective in differentiating the dynamics of particle hopping among bipartite and non-bipartite structures [1]. More results can be found in [34].

A graph-spectrum-based invariant, recently put forward by Estrada [27], is defined as

(7) $EE = EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$

We propose to call it the Estrada index. Although invented in year 2000 [12], the Estrada index has already found numerous applications. It was used to quantify the degree of folding of long-chain molecules, especially proteins [12, 13, 14]; for this purpose the $EE-$ values of pertinently constructed weighted graphs were employed. Another, fully unrelated, application of EE (this time of simple graphs, like those studied in the present paper) was put forward by Estrada and Rodriguez-Velazquez [15, 16]. They showed that EE provides a measure of the centrality of complex (communication, social, metabolic, etc.) networks. In addition to this, in a recent work [17] a connection between EE and the concept of extended atomic branching was considered. Until now only some elementary and easy general mathematical properties of the Estrada index were established. Of these, worth mentioning are only the following:

Specifically The adjacency matrix of a zero divisor graph $\Gamma(L)$ is defined as $A = [a_{il}]$, where

$a_{ij} = \{1, \ v_i \wedge v_j = 1, \quad v_i, v_j \in Z(L)^* \ 0, \quad otherwise$

Theorem 1.1. [16] *If the graph G is bipartite, and if n_0 is the multiplicity of its eigenvalue zero, then $EE(G) = n_0 + \sum_+ ch(\lambda_i)$ where ch stands for the hyperbolic cosine $\left(ch(x) = \frac{e^x+e^{-x}}{2}\right)$, whereas $\sum_+$ denotes summation over all positive eigenvalues of the corresponding graph.*

a) The Lattice $L = D_{p^n} \times D_{p^n}$, $n \geq 1$, where p is a prime number

The set $Z(L)^* = \{(1,p),(1,p^2),\cdots,(1,p^n),(p,1),(p^2,1),\cdots,(p^n,1)\}$. Here each vertex $(1.p^i)$ is adjacent to $(p^i,1)$ for all $i = 1,2,\ldots,n$ and the maximum degree of graph $\Delta(\Gamma(L)) = n$, since $\Gamma(L) \cong K_{n,n}$.

ISSN PRINT 2319 1775 Online 2320 7876

 UGC CARE Listed (Group -I) Journal Volume 11, Iss 12, 2022

Properties of adjacency matrix $A(\Gamma(D_{p^n} \times D_{p^n})), n \geq 1$:

(1) Determinant of adjacency matrix corresponding to zero divisor graph of $D_{p^n} \times D_{p^n}$ is 0

(2) Eigenvalues are $0(2n-2\ zeros)$, $-n$ and n

(3) Nullity of the adjacency matrix $A(\Gamma(D_{p^n} \times D_{p^n}))$ is $2n - 2$

(4) $E\big(\Gamma(L)\big) = |n| + |-n| = 2n$

(5) $ER\big(\Gamma(L)\big) = \frac{1}{2n-n} + \frac{1}{2n+n} + (2n-2)\frac{1}{2n} = \frac{3n+1}{n}$

(6) $\mathrm{EE}(\Gamma(L)) = e^{-n} + e^{n} + (2n-2)e^{0} = (2n-2) + e^{-n} + e^{n}$, by Theorem 1.1

(7) $EE_r\big(\Gamma(L)\big) = \frac{2n-1}{n-1} + \frac{2n-1}{3n-1} + (2n-2)$

(8) $H\big(\Gamma(L)\big) = (2n-2)e^{0} + e^{-n^2} + e^{n^2}$

(9) $M_k\big(\Gamma(L)\big) = n^k + (-n)^k,\ M_k = 0$ if k is odd and $M_k = 2n^k$ if k is even

b) The Lattice $L = D_{p^n q}, n \geq 1, p$ and q are distinct prime numbers

The set of zero divisor of $L = \{p, p^2, \ldots, p^n, q\}$ and the possible edges are $\{p, q\}$, $\{p^2, q\}, \ldots, \{p^n, q\}$ and the maximum degree of graph $\Delta(\Gamma(L)) = k$, since $\Gamma(L) \cong K_{1,n}$.

(1) Number of non - zero divisors in $\Gamma(D_{p^n q})$ are $n + 1$, therefore order of such adjacency matrix is $n + 1$.

(2) Eigenvalues are $0, 0, \ldots 0(n - 1\ zeros), \sqrt{n}$ and $-\sqrt{n}$

(3) Nullity of the adjacency matrix $A(\Gamma(D_{p^n q}))$ is $n - 1$

(4) $E\big(\Gamma(L)\big) = \big|\sqrt{n}\big| + \big|-\sqrt{n}\big| = 2\sqrt{n}$

(5) $ER\big(\Gamma(L)\big) = \frac{n-1}{n+1} + \frac{1}{(n+1)-\sqrt{n}} + \frac{1}{(n+1)+\sqrt{n}}$

(6) $\mathrm{EE}\big(\Gamma(L)\big) = e^{\sqrt{n}} + e^{-\sqrt{n}} + (n-1)e^{0} = e^{\sqrt{n}} + e^{-\sqrt{n}} + (n-1)$

(7) $EE_r\big(\Gamma(L)\big) = \frac{n}{n-\sqrt{n}} + \frac{n}{n+\sqrt{n}} + n - 1$

(8) $H\big(\Gamma(L)\big) = (n-1)e^{0} + e^{-\sqrt{n}^2} + e^{\sqrt{n}^2}$

(9) $M_k\big(\Gamma(L)\big) = \sqrt{n}^k + \big(-\sqrt{n}\big)^k$

c) The Lattice $L = D_{pqr}$, where p, q and r are distinct prime numbers

The set of zero divisors are $\{p, q, r, pq, qr, pr\}$ and the maximum degree of graph $\Delta(G) = 3$

The adjacency matrix corresponding to the zero divisor graph is

$$A\Big(\Gamma\big(D_{pqr}\big)\Big) = [0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0]$$

ISSN PRINT 2319 1775 Online 2320 7876

 UGC CARE Listed (Group -I) Journal Volume 11, Iss 12, 2022

Properties of adjacency matrix $A(\Gamma(D_{pqr}))$:

(1) Determinant of adjacency matrix corresponding to zero divisor graph of D_{pqr} is -1

(2) Eigenvalues are $-1.6180, -1.6180, -0.4142, 0.6180, 0.6180, 2.4142$(correct to four places of decimals)

(3) Nullity of the adjacency matrix $A(\Gamma(D_{pqr}))$is 0

(4) $E(\Gamma(L)) = 2|(-1.6180)| + 2|0.6180| + |2.4142| + |-0.4142| = 7.3004$

(5) $ER(\Gamma(L)) = 1.0689$

(6) $EE(\Gamma(L)) = 15.9489$

(7) $EE_r(\Gamma(L)) = 6.6502$

(8) $H(\Gamma(L)) = 29.6243$

(9) $Mk(\Gamma(L)) = 2(-1.6180)^k + 2(0.6180)^k + (-0.4142)^k + (2.4142)^k, M_0 = 6, M_1 = 1.2426, M_2 = 11.9996, M_3 = 6.0002$

d) The Lattice $L = D_{pqrs}$, where p, q, r and s are distinct prime numbers

The set of zero divisors are $\{p, q, r, s, pq, pr, ps, qr, qs, rs, pqr, pqs, prs, qrs\}$and the maximum degree of graph is $\Delta(\Gamma(L)) = 7$ and the zero divisor graph of D_{pqrs} is

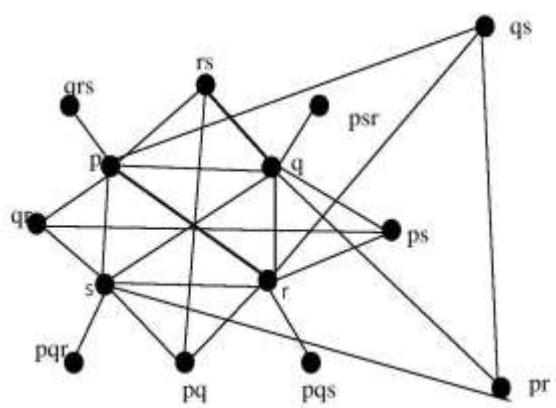

Figure 1. $\Gamma(D_{pqrs})$

The adjacency matrix corresponding to the zero divisor graph is partitioned in the form $A\left(\Gamma(D_{pqrs})\right) = [A\ B\ B^T\ 0\ ]$, where 0 denotes the zero matrix of order 7 and

$A =$

$[0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ ]$

and

$B = [1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ ]$

ISSN PRINT 2319 1775 Online 2320 7876

 UGC CARE Listed (Group -I) Journal Volume 11, Iss 12, 2022

Properties of adjacency matrix $A(\Gamma(D_{pqrs}))$:

(1) Determinant of adjacency matrix corresponding to zero divisor graph of D_{pqrs} is -1.

(2) Eigenvalues are $-2.6180, -2.6180, -2.6180, -1, -0.3820, -0.3820, -0.3820, 0.2087, 1, 1, 1, 1, 1, 4.7913$ (correct to four places of decimals).

(3) Nullity of the adjacency matrix $A\left(\Gamma\left(D_{pqrs}\right)\right)$ is 0

(4) $E(\Gamma(L)) = 3|-2.6180| + |-1| + 3|-0.3820| + |0.2087| + 5 + |4.7913| = 20$

(5) $ER(\Gamma(L)) = 1.0215$

(6) $EE(\Gamma(L)) = 137.92$

(7) $EE_r(\Gamma(L)) = 15.6161$

(8) $H(\Gamma(L)) = 5.7605$

(9) $M_k\left(\Gamma(L)\right) = 3(-2.6180)^k + (-1)^k + 3(-0.3820)^k + (0.2087)^k + 5 + (4.7913)^k$, $M_0 = 14, M_1 = 0, M_2 = 49.99966, M_3 = 60.003, M_4 = 673.998$

e) The Lattice $L = D_{pqrst}$, where p, q, r, s and t are distinct prime numbers

The set of zero divisors are $\{p, q, r, s, t, pq, pr, ps, pt, qr, qs, qt, rs, rt, st, pqr, pqs, pqt, qrs, qrt, qst, pst, rst, prs, prt, qrst, prst, pqst, pqrt, pqrs\}$ and the maximum degree of graph is $\Delta(\Gamma(L)) = 15$. The zero divisor graph of D_{pqrst} is

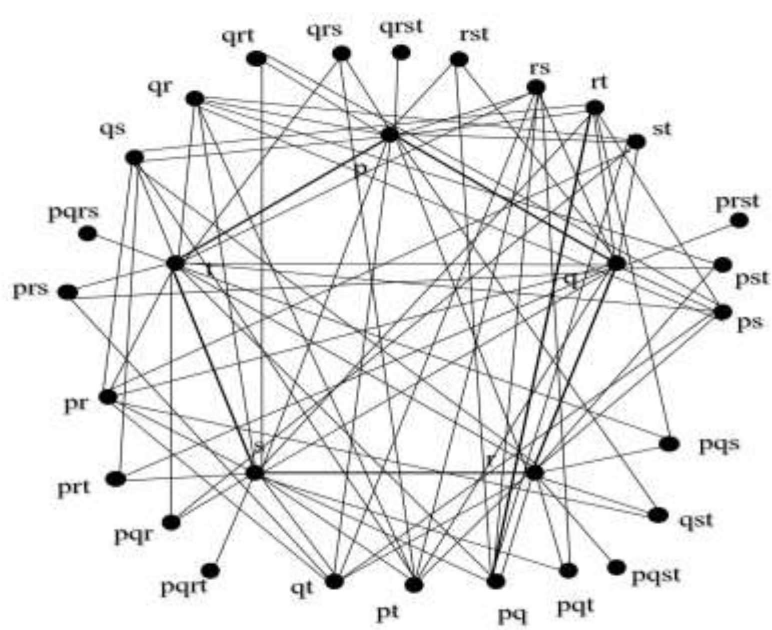

Figure 2. $\Gamma(D_{pqrst})$

ISSN PRINT 2319 1775 Online 2320 7876

 UGC CARE Listed (Group -I) Journal Volume 11, Iss 12, 2022

The adjacency matrix corresponding to the zero divisor graph, partitioned into the square matrix of order 10 $A\left((\Gamma(D_{pqrst})\right) = [A\ B\ C\ B^T\ D\ E\ C^T\ F\ 0\ ]$, where

$A =$
[0 1 1 1 1 0 0 0 0 1 1 0 1 1 1 0 1 1 1 0 1 1 0 1 1 1 0 1 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 0 1 1 1 0 1 0 0 1 1 1 0 0 0 0

B =
[1 1 1 1 1 0 0 0 1 1 0 0 1 1 1 0 0 0 1 0 1 1 0 0 1 0 1 1 0 0 0 1 0 1 0 1 0 1 0 1 1 0 1 0 0 1 1 0 1 0 0 0 1 1 1 0 0 0 0 0

C
= [1 0 1 0 0 1 0 0 0 0 0 1 1 1 1 0 1 0 0 0 1 1 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 0 1 0 0 1 0 0 0

$D = [d_{ij}]$, where
$\{1, \ \ for\ d_{14}\ , d_{23}\ ,$
$d_{32}\ , d_{38}, d_{41}\ , d_{47}, d_{56}, d_{65}, d_{74}, d_{73}\ \ 0\ ,$ $\qquad otherwise$

$E = [e_{ij}]$, where $e_{ij} = \{1, \ \ for\ e_{15}, \ \ e_{24}\ 0, \ \ otherwise$ and $F = [f_{ij}]$, where $f_{ij} =$
$\{1, \ \ for\ f_{42} \qquad 0, \ \ otherwise$

Properties of adjacency matrix $A(\Gamma(D_{pqrst}))$:

(1) Determinant of adjacency matrix corresponding to zero divisor graph of D_{pqrst} is 0

(2) Eigenvalues are $-4.3518, -4.2361, -4.0815, -2.2437, -4.1474, -0.8441, -0.6180$(repeated 6 times) $, -0.1814, -0.4346, -0.0981, 0, 0.2020, 0.2361, 0.2993, 0.5269,$ 1.2979,1.3289,1.6180(repeated 6 times),1.9123, 8.8153(correct to four places of decimals)

(3) Nullity of the adjacency matrix $A(\Gamma(D_{pqrst}))$is 1

(4) $E(\Gamma(L)) = |-4.3518| + |-4.2361| + |-4.0815| + |-2.2437| + |4.1474| + |-0.8441| + 6|-0.6180| + |-0.1814| + |-0.4346| + |-0.0981| + |0.2020| + 2|0.2361| + |0.2993| + |0.5269| + |1.2979| + |1.3289| + 6|1.6180| + |1.9123| + |8.8153| = 48.889500$

(5) $ER(\Gamma(L)) = 3.2408057$

(6) $EE(\Gamma(L)) = 6793.7446$

(7) $EE_r(\Gamma(L)) = 29.371103$

(8) $H(\Gamma(L)) = 12.775930$

(9) $M_k(\Gamma(L)) = (-4.3518)^k + (-4.2361)^k + (-4.0815)^k + (-2.2437)^k + (-.1474)^k + (-0.8441)^k + 6(-0.6180)^k + (-0.1814)^k + (-0.4346)^k + (-0.0981)^k + (0.2020)^k + (0.2361)^k + (0.2993)^k + (0.5269)^k + (1.2979)^k + (1.3289)^k + 6(1.6180)^k + (1.9123)^k + (8.8153)^k, M_0 = 30, M_1 = 0, M_2 = 180.05588, M_3 = 411.01823, \ M_4 = 7380.1176$

Generalization of the results from analysis over the lattice $D_{p_1p_2\cdots p_n}$

(1) Number of non –zero zero divisors in $\Gamma(D_{p_1p_2\cdots p_n})$ is $2^n - 2$. Therefore order of such adjacency matrix is $2^n - 2 \times 2^n - 2$

(2) $\Delta(G) = \frac{2^{n}-2}{2} = 2^{n-1} - 1$

f) The lattice $L = D_{pq^2 r^3}$, where p, q and r are distinct prime numbers

$V\left(\Gamma(L)\right) = \{q, q^2, pq, pq^2, r, r^2, r^3, pr, pr^2, pr^3, p, qr^3, qr^2, qr, q^2r, q^2r^2, q^2r^3\}$ and $\Delta(\Gamma(L)) = 11$. The zero divisor graph of $D_{pq^2 r^3}$ is

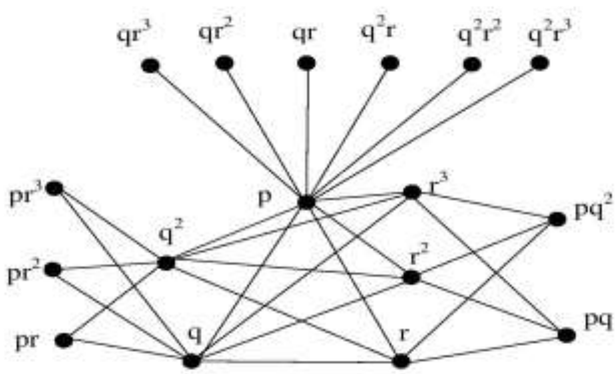

Figure 3. $\Gamma(D_{pq^2 r^3})$

The adjacency matrix corresponding to the zero divisor graph partitioned in the form

$A\left(\Gamma\left(D_{pq^2 r^3}\right)\right) = [O_{4\times4}\ P_{4\times7}\ \ O_{4\times6}\ Q_{4\times4}\ R_{4\times7}\ O_{4\times6}\ S_{4\times4}\ T_{4\times7}\ U_{4\times6}\ O_{5\times4}\ V_{5\times7}\ O_{5\times6}\ ]$, where

$Q = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ ]$,

$T = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ ]$,

$P = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ ]$

R is the block matrix whose entries are zero, except the last column ($[1, 1, 1, 0]$). S is the block matrix whose entries are zero, except first two columns, whose entries are 1, 1, 1, 0. U, whose entries are zero except third row (all entries in this row is equal to 1), and V , whose entries are zero except last column (all entries in this row is equal to 1), and O is the zero matrix.

Properties of adjacency matrix $A(\Gamma(D_{pq^2 r^3}))$:

(1) Determinant of adjacency matrix corresponding to zero divisor graph of $D_{pq^2 r^3}$ is 0

(2) Eigenvalues are $-3.9881, -3.1735, -1.2065, 0$(repeated 11 times),1.5045, 1.8907,4.9730(correct to four places of decimals)

(3) Nullity of the adjacency matrix $A(\Gamma(D_{pq^2 r^3}))$ is 11

(4) $E(\Gamma(L)) = |-3.9881| + |-3.1735| + |-1.2065| + |1.5045| + |1.8907| + |4.9730| = 16.7363$

(5) $ER(\Gamma(L)) = 1.0130658$

 UGC CARE Listed (Group -I) Journal Volume 11, Iss 12, 2022

(6) $EE(\Gamma(L)) = 247.77053$
(7) $EE_r(\Gamma(L)) = 17.253621$
(8) $H(\Gamma(L)) = 11.365299$
(9) $M_k(\Gamma(L)) = (-3.9881)^k + (-3.1735)^k + (-1.2065)^k + (1.5045)^k + (1.8907)^k + (4.9730)^k, M_0 = 6, M_1 = 0.0001000, M_2 = 58.000682, M_3 = 36.002782, M_4 = 986.02446$

g) The lattice $L = D_{pq^3 r^4}$, where p, q and r are distinct prime numbers

$V(\Gamma(L)) = \{q, q^2, q^3, pq, pq^2, pq^3, r, r^2, r^3, r^4, pr, pr^2, pr^3, pr^4, p, qr^4, qr^3, qr^2, qr, q^2r, q^2r^2, q^2r^3, q^2r^4, q^3r, q^3r^2\}$ and $\Delta(\Gamma(L)) = 19$. The adjacency matrix corresponding to the zero divisor graph is partitioned in the form

$A\left(\Gamma\left(D_{pq^2 r^3}\right)\right) = [A_{15\times15}\ B_{15\times12}\ C_{12\times15}\ O_{12\times12}]$, where

$A = [0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0$

B, is the block matrix whose entries are zero, except the last row(all entries in this row is equal to 1) and C, is the block matrix whose entries are zero, except the last column(all entries in this row is equal to 1) and $O_{12\times12}$ is a zero matrix.

Properties of adjacency matrix $A\left(\Gamma\left(D_{pq^3 r^4}\right)\right)$:

(1) Determinant of adjacency matrix corresponding to zero divisor graph of $D_{pq^3 r^4}$ is 0
(2) Eigenvalues are $-5.6155, -4.2407, -1.7951, 0$(repeated 21 times), 2.1369,2.8297,6.6847(correct to four places of decimals)
(3) Nullity of the adjacency matrix $A\left(\Gamma\left(D_{pq^3 r^4}\right)\right)$is 21
(4) $E(\Gamma(L)) = |-5.6155| + |-4.2407| + |-1.7951| + |2.1369| + |2.8297| + |6.6847| = 23.302600$
(5) $ER(\Gamma(L)) = 1.0059930$
(6) $EE(\Gamma(L)) = 846.66828$
(7) $EE_r(\Gamma(L)) = 27.175324$
(8) $H(\Gamma(L)) = 21.050589$
(9) $M_k(\Gamma(L)) = (-5.6155)^k + (-4.2407)^k + (-1.7951)^k + (2.1369)^k + (2.8297)^k + (6.6847)^k, M_0 = 6, M_1 = 0, M_2 = 109.99852, M_3 = 71.997481, M_4 = 3409.9095$

h) The lattice $L = D_{(pqr)^2}$, where p, q and r are distinct prime numbers

$V(\Gamma(L)) = \{p, p^2, q, q^2, r, r^2, qr, q^2r, qr^2, q^2r^2, pr, p^2r, pr^2, p^2r^2, pq, p^2q, pq^2, p^2q^2\}$ and $\Delta(G) =$

ISSN PRINT 2319 1775 Online 2320 7876

Research paper © 2012 IJFANS. All Rights Reserved, UGC CARE Listed (Group -I) Journal Volume 11, Iss 12, 2022

6. The adjacency matrix corresponding to the zero divisor graph, partitioned into block square matrices of order 9.

$A\left(\Gamma\left(D_{(pq}\right.\right.$

$X =$
[0 0 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 0 0 1 1 0 0 0 1 1 0 0 1 1 0 0 0 1 1 1 1 0 0 0 0 0 1 1 1 1 0 0 0 0 0 1 1 0 0 0
,
$Y =$
[1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 1 1 1 1 0 0 0 0 0

Properties of adjacency matrix $A\left(\Gamma\left(D_{(pqr)^2}\right)\right)$:

(1) Determinant of adjacency matrix corresponding to zero divisor graph of $D_{(pqr)^2}$ is 0
(2) Eigenvalues are $-4, -4, 0$(repeated 13 times), $2, 2, 5.4641$(correct to four decimal places)
(3) Nullity of the adjacency matrix $A\left(\Gamma\left(D_{(pqr)^2}\right)\right)$ is 13
(4) $E(\Gamma(L)) = |-4| + |-4| + |2| + |2| + |5.4641| = 17.4641$
(5) $ER(\Gamma(L)) = 0.2956800$
(6) $EE(\Gamma(L)) = 263.87805$
(7) $EEr(\Gamma(L)) = 5.3593748$
(8) $H(\Gamma(L)) = 13.036632$
(9) $M_k(\Gamma(L)) = 2(-4)^k + 2(2)^k + (5.4641)^k, M_0 = 5, M_1 = 1.4641, M_2 = 69.8563888, M_3 = 51.1382941, M_4 = 1435.40395$

References

[1] Alhomaidhi, A.A.; Al-Thukair, F.; Estrada, E. *Gaussianization of the spectra of graphs and networks*. Theory and applications. J. Math. Anal. Appl. 2019, 470, 876- 897. [CrossRef]
[2] M. Benzi, P. Boito, *Quadrature rule-based bounds for functions of adjacency matrices*, Lin. Algebra Appl. 433 (2010) 637-652.
[3] Bhat V.K, Ravi Raina, *A Note on Zero-divisor Graph over Rings*, Int. J. Contemp. Math. Sci. 2(14), 667-671(2007).
[4] B. Bollobos and I. Rival, *The maximal size of the covering graph of a lattice*, Algebra Universalis **9** (1979), 371-373.
[5] G. Chartrand and P. Zhang, *Introduction to Graph theory*, Wadsworth and Brooks/Cole, Monterey, CA 1986.
[6] D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge Univ. Press, Cambridge, 2010.

[7] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs* - Theory and Application, Academic Press, New York, 1980.

[8] X. Chen, J. Qian, *Bounding the resolvent Estrada index of a graph*, J. Math. Study 45 (2012) 159-166.

[9] X. Chen, J. Qian, *On resolvent Estrada index*, MATCH Commun. Math. Comput.Chem. 73 (2015) 163-174.

[10] D. Duffus and I. Rival, *Path length in the covering graph of a lattice*, Discrete Math. **19** (1977), 139-158.

[11] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press Cambridge 2002.

[12] E. Estrada, *Characterization of 3D molecular structure*, Chem. Phys. Lett. 319 (2000) 713-718.

[13] E. Estrada, *Characterization of the folding degree of proteins*, Bioinformatics 18 (2002) 697-704.

[14] E. Estrada, *Characterization of the amino acid contribution to the folding degree of proteins*, Proteins 54 (2004) 727-737.

[15] E. Estrada, J.A. Rodríguez-Velázquez, *Subgraph centrality in complex networks*, Phys. Rev. E 71 (2005) 056103-1- 056103-9.

[16] E. Estrada, J.A. Rodríguez-Velázquez, *Spectral measures of bipartivity in complex networks*, Phys. Rev. E 72 (2005) 046105-1-046105-6.

[17] E. Estrada, J.A. Rodríguez-Velázquez, M. Randic, *Atomic branching in molecules*, Int. J. Quantum Chem. 106 (2006)823-832

[18] Estrada, E.; Alhomaidhi, A.A.; Al-Thukair, F. "*Exploring the Middel Earth" of network spectra via a Gaussian matrix function*. Chaos 2017, 27, 023-109.

[19] E. Estaji and K. Khashyarmanesh, *The zero divisor graph of a lattice*, Results Math. **61** (2012) 1-11.
[20] I. Gutman, *The energy of a graph*, Ber. Math.-Statist. Sekt. Forschungsz. Graz. 103 (1978) 1-22.

[21] I. Gutman, *Comparative studies of graph energies*, Bull. Acad. Serbe Sci. Arts (Cl.Sci. Math. Nature.) 144 (2012) 1-17.

[22] I. Gutman, *Census of graph energies*, MATCH Commun. Math. Comput. Chem. 74(2015) 219-221.

[23] I. Gutman, B. Furtula, X. Chen, J. Qian, *Graphs with smallest resolvent Estrada indices*, MATCH Commun. Math. Comput. Chem. 73 (2015) 267-270.

[24] I. Gutman, B. Furtula, X. Chen, J. Qian, *Resolvent Estrada index* computational and mathematical studies, MATCH Commun. Math. Comput. Chem. 74 (2015) 431-440.

[25] I. Gutman, B. Furtula, E. Zogi´*c*, E. Glogić, *Resolvent energy*, MATCH Commun. Math. Comput. Chem. 75 (2016) 279-290.

[26] I. Gutman, B. Furtula, E. Zogić, E. Glogić, *Resolvent energy*, in: I. Gutman, X. Li (Eds.), Graph Energies - Theory and Applications, Univ. Kragujevac, Kragujevac, 2016, pp. 277-290.

[27] I. Gutman, X. Li (Eds.), *Energies of Graphs* - Theory and Applications, Univ. Kragujevac, Kragujevac, 2016.

[28] E. Estrada, D. J. Higham, Network properties revealed through matrix functions, SIAM Rev. 52 (2010) 696-714.

[29] E. Gedeonová, *Lattices, whose covering graphs are S-graphs*, Colloq Math. Soc. János Bolyai **33** (1980), 407-435.

[30] X. Li. Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[31] V. Nikiforov, *The energy of graphs and matrices*, J. Math. Anal. Appl. 326 (2007) 1472-1475.

[32] V. Nikiforov, *Energy of matrices and norms of graphs*, in: I. Gutman, X. Li (Eds.), Graph Energies Theory and Applications, Univ. Kragujevac, Kragujevac, 2016, pp. 5-48.

[33] S. K. Nimbhorkar, M. P. Wasadikar and M. M. Pawar, *Coloring of lattices*, Math. Slovaca **60** (2010) 419-434.

[34] 28. Shang, Y. *Lower bounds for Gaussian Estrada index of graphs*. Symmetry 2018, 10, 325. [CrossRef]

[35] Wang, L.W.; Zunger, A. *Solving Schrödinger's equation around a desired energy*: Application to silico quantum dots. J. Chem. Phys. 1994, 100, 2394. [CrossRef]