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Approximate Analytic Solution for Harmonic Oscillator with Fractional Order Damping Term

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Abstract. In this paper, analytical and numerical results are reported on the approximate solution for harmonic oscillator equation with fractional order damping term by Adomian Decomposition Method (ADM). The ADM solution is then compared with the well known series solution of the problem. It is found that the results lead to an excellent agreement. The graphical representation of the solution has been presented for different values of damping coefficient and frequency of the oscillator.

Keywords: ADM, Time fractional damping term, Harmonic oscillator, Power series method.

INTRODUCTION

Recently a great deal of interest has been focused on Adomian's Decomposition Method (ADM) and its applications to a wide class of physical Problems containing fractional derivatives [7,8,10,13,14].The ADM employed here is adequately discussed in published literature [5,6,15], but it still deserves emphasis to point out the very significant advantages over other methods. The said method can also be an effective procedure for the solution of Harmonic oscillator equation with fractional order damping term. The fractional differential equation have been used to model many physical and engineering processes such as frequency dependent damping behavior of materials, motion of a large thin plate in a Newtonian fluid, creep and relaxation function for viscoelastic materials etc [11,12,16]. Moreover, phenomena in electromagnetic, acoustics, viscoplasticity, and electrochemistry are also described by differential equations of fractional order [4,9,16].

In the present paper, Adomian decomposition method (ADM) is implemented to the harmonic oscillator equation with a fractional order damping term. In these schemes the solution constructed in power series with easily computable components. The ADM solution is then compared with the well known series solution of the problem. It is found that the result leads to an excellent agreement. The numerical calculations are carried out and are presented graphically to examine its shape.

MATHEMATICAL ASPECTS OF FRACTIONAL CALCULUS

Many definitions of fractional calculus are used to solve the problems of fractional differential equations. The most frequently encountered definitions include Riemann-Liouville, Caputo, Weyl and Riesz fractional operator. We introduce the following definitions [1,2].

DEFINITION

Let $\alpha \in R^+$.The integral operator I^α defined on the usual Lebesgue space $L(a,b)$ by

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$$I^{\alpha} f(x)=\frac{d^{-\alpha}}{dx^{-\alpha}} f(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1} f(t)dt, \tag{1}$$

for $a \le x \le b$,is called Riemann-Liouville fractional integral operator of order $\alpha > 0$.

DEFINITION

The Riemann-Liouville definition of fractional order derivative is

$$D^{\alpha} f(x)=\frac{d^{\alpha}}{dx^{\alpha}} f(x)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_{0}^{x}(x-t)^{m-\alpha-1} f(t)dt \tag{2}$$

where m is an integer that satisfies $m-1 \le \alpha < m$.

DEFINITION

A modified fractional differential operator D^{α} proposed by Caputo is given by

$$D^{\alpha} f(x)=\frac{d^{\alpha}}{dx^{\alpha}} f(x)=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-t)^{m-\alpha-1} f^{m}(t)dt \tag{3}$$

where $\alpha \in R^{+}$,is the order of operation and m is an integer that satisfies $m-1 \le \alpha < m$.

ANALYSIS OF THE METHOD

In this section we consider the following equation

$$\frac{d^2 x(t)}{dt^2}+\omega^2 x(t)+2k\frac{d^{1/2}x(t)}{dt^{1/2}}=0 \tag{4}$$

with appropriate initial and boundary conditions. The standard form of the eq. (3) in an operator form [5,6] is given below:

$$Lx(t)+\omega^2 x(t)+2kD^{1/2}x(t)=0\,, \tag{5}$$

where the operators are $L \equiv \frac{d^2}{dt^2}$, $D^{1/2}=\frac{d^{1/2}}{dt^{1/2}}$, k is the damping coefficient and is the ω frequency of the oscillation. Operating with the integral operator L^{-1} (assuming the integral exists), we have

$$x(t)=a+bt-\omega^2 L^{-1}[x(t)]-2kL^{-1}\left[D^{1/2}x(t)\right] \tag{6}$$

where a, b are constants, to be determined from initial and boundary conditions, following the analysis of ADM [5,6] ,we expect the decomposition of the solution into a sum of components to be defined by the following decomposition series form:

$$x(t)=\sum_{n=0}^{\infty} x_n(t). \tag{7}$$

Hence eq. (3) can be written as,

$$\sum_{n=0}^{\infty} x_n(t)=a+bt-\omega^2 L^{-1}\left[\sum_{n=0}^{\infty} x_n(t)\right]-2kL^{-1}\left[D^{1/2}\sum_{n=0}^{\infty} x_n(t)\right] \tag{8}$$

Identifying the zero component $x_0(t)$ by $x_0(t)=a+bt,$ the remaining components can be determined by using the following recurrence relation [5,6]:

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$$x_{n+1}(t)=-\omega^2L^{-1}[x_n(t)]-2kL^{-1}\left[D^{1/2}x_n(t)\right],n\geq 0$$

Consequently, we find that

$$x_1(t)=-a\left\{\frac{2}{\Gamma(5/2)}kt^{3/2}+\frac{1}{2!}\omega^2t^2\right\}-b\left\{\frac{2}{\Gamma(7/2)}kt^{5/2}+\frac{1}{3!}\omega^2t^3\right\},$$

$$x_2(t)=a\left\{\frac{4}{\Gamma(9/2)}\omega^2kt^{7/2}+\frac{8}{3!}k^3t^3+\frac{1}{4!}\omega^4t^4\right\}+b\left\{\frac{4}{\Gamma(11/2)}\omega^2kt^{9/2}+\frac{4}{4!}k^2t^4+\frac{1}{5!}\omega^4t^5\right\},$$

$$x_3(t)=-a\left\{\frac{8}{\Gamma(11/2)}k^3t^{9/2}+\frac{6}{\Gamma(13/2)}\omega^4kt^{11/2}+\frac{12}{5!}\omega^2k^2t^5+\frac{1}{6!}\omega^6t^6\right\}$$

$$-b\left\{\frac{8}{\Gamma(13/2)}k^3t^{11/2}+\frac{6}{\Gamma(15/2)}\omega^4kt^{13/2}+\frac{12}{6!}\omega^2k^2t^6+\frac{1}{7!}\omega^6t^7\right\},$$

$$x_4(t)=a\left\{\frac{32}{\Gamma(15/2)}\omega^2k^3t^{13/2}+\frac{8}{\Gamma(17/2)}\omega^6kt^{15/2}+\frac{16}{6!}k^4t^6+\frac{24}{7!}\omega^4k^2t^7+\frac{1}{8!}\omega^8t^8\right\}$$

$$-b\left\{\frac{32}{\Gamma(17/2)}\omega^2k^3t^{15/2}+\frac{8}{\Gamma(19/2)}\omega^6kt^{17/2}+\frac{16}{7!}k^4t^7+\frac{24}{8!}\omega^4k^2t^8+\frac{1}{9!}\omega^8t^9\right\},$$

$$x_5(t)=-a\left\{\frac{32}{\Gamma(17/2)}k^5t^{15/2}+\frac{80}{\Gamma(19/2)}\omega^4k^3t^{17/2}+\frac{10}{\Gamma(21/2)}\omega^8kt^{19/2}+\frac{80}{8!}\omega^2k^4t^8+\frac{40}{9!}\omega^6k^2t^9+\frac{1}{10!}\omega^{10}t^{10}\right\}$$

$$-b\left\{\frac{32}{\Gamma(19/2)}k^5t^{17/2}+\frac{80}{\Gamma(21/2)}\omega^4k^3t^{19/2}+\frac{10}{\Gamma(23/2)}\omega^8kt^{21/2}+\frac{80}{9!}\omega^2k^4t^9+\frac{40}{10!}\omega^6k^2t^{10}+\frac{1}{11!}\omega^{10}t^{11}\right\}$$

and so on.

Therefore, all components of $x(t)$ are calculable and from (4), the solution is completely determined.

The expression $\phi_n(t)=\sum_{i=0}^{n-1}x_i(t)$ is the $n-$term approximation to x. Here it is to be noted that the decomposition series solution is very rapidly convergent [6,7,8,14] and only a few terms of the series solution leads to a very good approximation with the actual solution of the problem. Generally, only a few terms are sufficient for most purpose and we can proceed further with little effort [6,7,8,18].

Therefore, the solution is

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$$x(t)=a\left\{1-\left[\frac{2}{\Gamma\left(5/2\right)}kt^{3/2}+\frac{1}{2!}\omega^2t^2\right]+\left[\frac{4}{3!}k^2t^3+\frac{4}{\Gamma\left(9/2\right)}\omega^2kt^{7/2}+\frac{1}{4!}\omega^4t^4\right]-\right.$$
$$\left.\left[\frac{8}{\Gamma\left(11/2\right)}k^3t^{9/2}+\frac{12}{5!}\omega^2k^2t^5+\frac{6}{\Gamma\left(13/2\right)}\omega^4kt^{11/2}+\frac{1}{6!}\omega^6t^6\right]+\ldots\right\}$$

$$+b\left\{t-\left[\frac{2}{\Gamma\left(7/2\right)}kt^{5/2}+\frac{1}{3!}\omega^2t^3\right]+\left[\frac{4}{4!}k^2t^4+\frac{4}{\Gamma\left(11/2\right)}\omega^2kt^{9/2}+\frac{1}{5!}\omega^4t^5\right]-\right.$$
$$\left.\left[\frac{8}{\Gamma\left(13/2\right)}k^3t^{11/2}+\frac{12}{6!}\omega^2k^2t^6+\frac{6}{\Gamma\left(15/2\right)}\omega^4kt^{13/2}+\frac{1}{7!}\omega^6t^7\right]+\ldots\right\} \tag{9}$$

It can be written as follows:

$$x(t)=a\left\{\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\left[\sum_{i=0}^{\infty}(-2k)^i\frac{{}^{j+i}C_i\,t^{2j+\frac{3}{2}i}}{\Gamma\left(1+2j+\frac{3}{2}i\right)}\right]\right\}$$
$$+\frac{b}{\omega}\left\{\sum_{j=0}^{\infty}(-1)^j\omega^{2j+1}\left[\sum_{i=0}^{\infty}(-2k)^i\frac{{}^{j+i}C_i\,t^{1+2j+\frac{3}{2}i}}{\Gamma\left(2+2j+\frac{3}{2}i\right)}\right]\right\}.$$

The decomposition series solution generally converges very rapidly in real physical problems [1,2,6,7,8,14]. It is also worth noting that a rapid stabilization to an acceptable accuracy is evident, when numerical computation of the analytic approximation is carried out [1,5,6,17].

VERIFICATION BY POWER SERIES METHOD

In view of the fractional differential eq. (1), we can take the solution in the form of the following fractional power series [1,2]:

$$x(t)=\sum_{n=0}^{\infty}P_nt^{n/2}. \tag{10}$$

Substituting this expression (1) into eq. (2), we get

$$\sum_{n=1}^{\infty}\frac{n}{2}\left(\frac{n}{2}-1\right)P_nt^{n/2-2}+\omega^2\sum_{n=0}^{\infty}P_nt^{n/2}+2k\sum_{n=0}^{\infty}P_nt^{(n-1)/2}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}=0. \tag{11}$$

By equating the coefficients of different powers of t, we obtain (assuming $P_0=a$ and $P_2=b$)

$P_1=0,$

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$$P_3 = -\frac{2}{\Gamma\left(5/2\right)} ka,$$

$$P_4 = -\frac{1}{2!} a\omega^2,$$

$$P_5 = -\frac{2}{\Gamma\left(7/2\right)} kb,$$

$$P_6 = \frac{4}{3!} ak^2 - \frac{1}{3!} b\omega^2,$$

$$P_7 = \frac{4}{\Gamma\left(9/2\right)} ak\omega^2,$$

$$P_8 = \frac{1}{4!} a\omega^4 + \frac{4}{4!} bk^2,$$

$$P_9 = \frac{4}{\Gamma\left(11/2\right)} bk\omega^2 - \frac{8}{\Gamma\left(11/2\right)} ak^3,$$

$$P_{10} = \frac{1}{5!} b\omega^4 - \frac{12}{4!} ak^2\omega^2,$$

$$P_{11} = -\frac{6}{\Gamma\left(13/2\right)} ak\omega^4 - \frac{8}{\Gamma\left(13/2\right)} bk^3,$$

$$P_{12} = -\frac{12}{6!} bk^2\omega^2 + \frac{16}{6!} ak^4 - \frac{1}{6!} a\omega^4,$$

$$P_{13} = -\frac{6}{\Gamma\left(15/2\right)} bk\omega^4 + \frac{32}{\Gamma\left(15/2\right)} ak^3\omega^2,$$

$$P_{14} = \frac{24}{7!} ak^2\omega^4 + \frac{16}{7!} bk^4 - \frac{1}{7!} b\omega^4$$

and so on.

The rest of the terms can be calculated in a similar manner. Therefore, the solution of eq. (5) is

$$x(t) = a\left\{1 - \frac{2}{\Gamma\left(5/2\right)} kt^{3/2} - \frac{1}{2!}\omega^2 t^2 + \frac{4}{3!} k^2 t^3 + \frac{4}{\Gamma\left(9/2\right)} k\omega^2 t^{7/2} + \frac{1}{4!}\omega^4 t^4 - \frac{8}{\Gamma\left(11/2\right)} k^3 t^{9/2} - \right.$$

$$\left. \frac{12}{5!} k^2\omega^2 t^5 - \frac{6}{\Gamma\left(13/2\right)} k\omega^4 t^{11/2} + \ldots\right\} + \frac{b}{\omega}\left\{\omega t - \frac{2}{\Gamma\left(7/2\right)} k\omega t^{5/2} - \frac{1}{3!}\omega^3 t^3 + \frac{4}{4!}\omega k^2 t^4 + \right.$$

$$\left. \frac{4}{\Gamma\left(11/2\right)} \omega^3 k t^{9/2} + \frac{1}{5!}\omega^5 t^5 - \frac{8}{\Gamma\left(13/2\right)} \omega k^3 t^{11/2} + \ldots\right\}. \tag{12}$$

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This solution is same as the solution (8), obtained by ADM.

NUMERICAL COMPUTATION AND GRAPHICAL REPRESENTATION

For numerical computation, the infinite series solution (9) is considered up to the term t^{22}, and the solution is represented graphically by (Fig.1, to Fig.6) together with Tables (Table 1 to Table 6), for different values of the parameters a, b and ω. The damping coefficient k has been considered for small values of $k = 0, 0.01, 0.02, 0.03, 0.04, 0.05$ respectively. Again, by keeping a, b and k fixed and for different values of $\omega(\omega = 4.0, 4.2, 4.4, 4.6, 4.8, 5.0)$, another set of results (Fig.7 to Fig.12) together with the Tables (Table 7 to Table 12) has been presented. The numerical results have been presented graphically to examine the shape of the resulting solution. From the graphs, it follows that the motions are oscillatory in nature as expected. Moreover, amplitude of the oscillation decreases with the increasing values of damping coefficient and time. It is interesting to note further that for a certain set of values of the parameters, the solution diminishes continuously after a certain period of time. The graphical representations of the ADM solution for different values of damping coefficient k and frequency of oscillation ω are given below:

Graphical representation of Adomian solution for different values of damping coefficient k

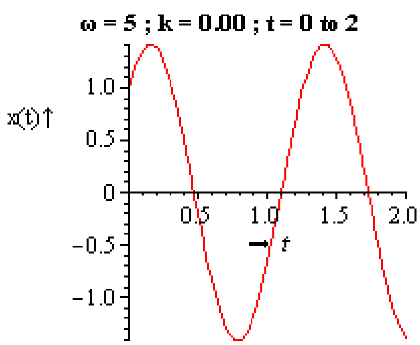

Fig. 1

Table-1: $k = 0.00$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.49315059
0.8	-1.41044611
1.2	0.68075478
1.6	0.84385821
2.0	-1.38309116

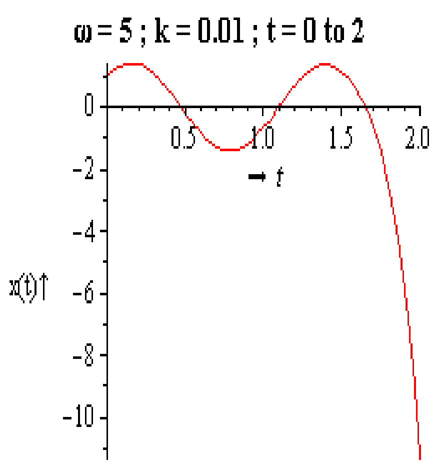

Fig. 2

Table-2: $k = 0.01$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.49033388
0.8	-1.40611187
1.2	0.67926174
1.6	0.50851231
2.0	-11.35333953

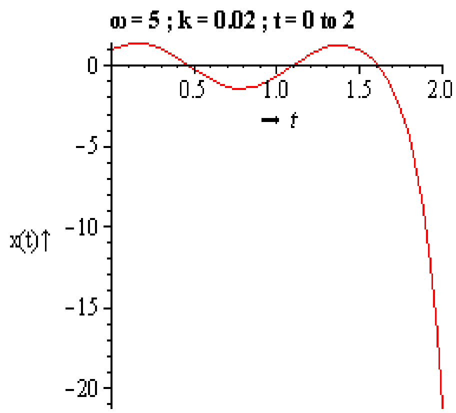

Fig.3

Table-3: $k = 0.02$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.48752425
0.8	-1.40177993
1.2	0.67773998
1.6	0.17492821
2.0	-21.25076849

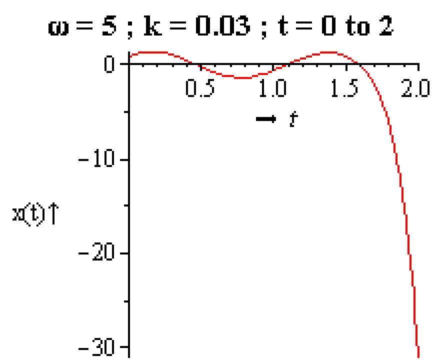

Fig.4

Table-4: $k = 0.03$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.48472169
0.8	-1.39745035
1.2	0.67618974
1.6	-0.15690289
2.0	-31.07588600

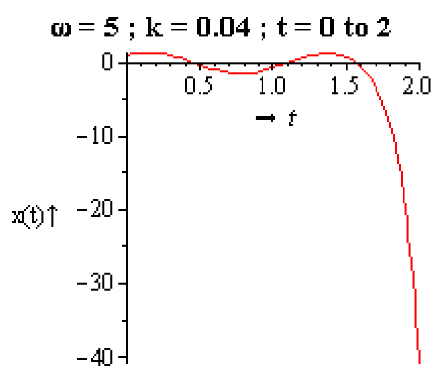

Fig.5

Table-5: $k = 0.04$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.48192617
0.8	-1.39312319
1.2	0.67461124
1.6	-0.48698973
2.0	-40.82921018

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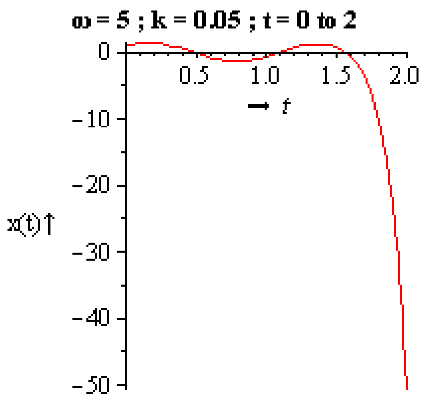

Fig. 6

Table-6: $k = 0.05$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.47913770
0.8	-1.38879850
1.2	0.67300473
1.6	-0.81534098
2.0	-50.51125510

Graphical representation of the Adomian method solution for different values of ω

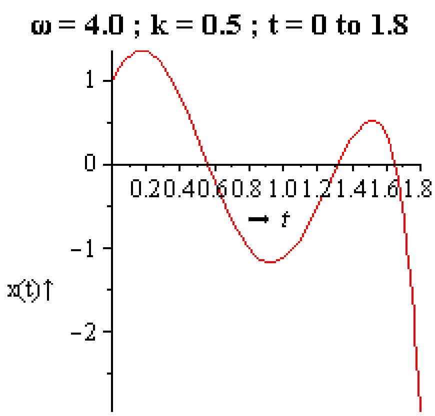

Fig. 7

Table-7: $\omega = 4.0$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.79908969
0.8	-1.01313611
1.2	-0.50674726
1.6	0.34138616
2.0	-18.79113420

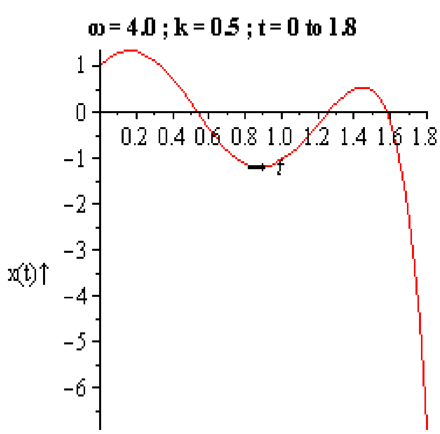

Fig. 8

Table-8: $\omega = 4.2$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.72066737
0.8	-1.10583428
1.2	-0.27092056
1.6	-0.20545060
2.0	-37.65257190

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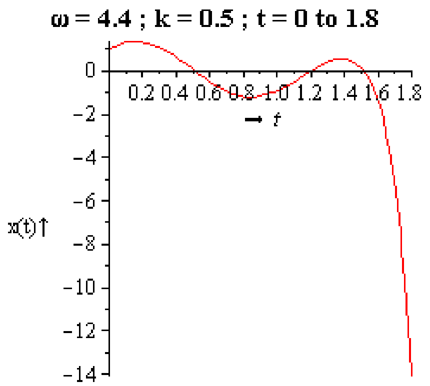

Fig. 9

Table-9: $\omega = 4.4$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.63705860
0.8	-1.17231091
1.2	-0.02363915
1.6	-1.38364484
2.0	-72.06142590

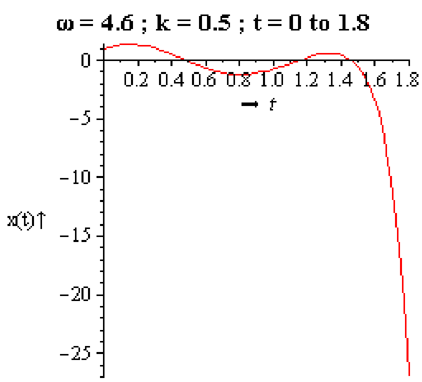

Fig. 10

Table-10: $\omega = 4.6$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.54881062
0.8	-1.21050817
1.2	-0.21577999
1.6	-3.58311855
2.0	-133.2679332

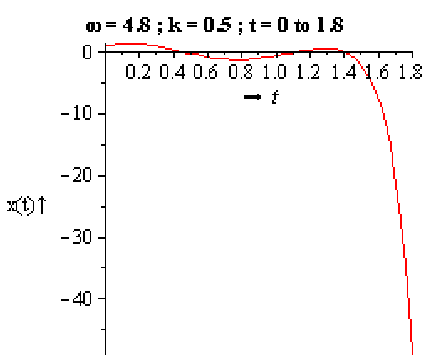

Fig. 11

Table-11: $\omega = 4.8$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.45650111
0.8	-1.21911096
1.2	0.42449808
1.6	-7.43131765
2.0	-239.5801426

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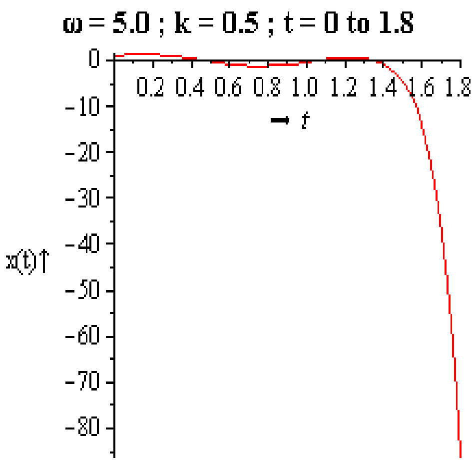

Fig. 12

Table-12: $\omega = 5.0$	
Value of t	Value of $x(t)$
0.0	1.00000000
0.4	0.36073445
0.8	-1.19758768
1.2	0.57532164
1.6	-13.92837653
2.0	-420.1051290

CONCLUSION

The present problem deals with the harmonic oscillator equation with fractional order damping term. It has been solved by ADM and the resulting solution is compared with that obtained by Power Series solution method. The results obtained by these methods are then compared numerically, through graphs and tables. The graphical representation of the Adomian solution has also been presented for different values of damping coefficient k and frequency ω. The advantage of this global methodology lies in the fact that it not only leads to an analytical continuous approximation which is very rapidly convergent but also shows the dependence giving insight into the character and behavior of the solution just as in a closed form solution. The ADM is straight forward and a rapid stabilization to an acceptable accuracy is evident when numerical computation of the analytic approximation is carried out. The present analysis exhibits the applicability of ADM to solve harmonic oscillator equation with a time fractional damping term. Moreover, this method does not require any transformation techniques, linearization or discretization of the variables and it does not make closure approximation or smallness assumptions. In fact, since the decomposition method, is not, in general, a perturbative method, it follows that the operator equation can be handled and accurate approximated solutions may be obtained for many physical problems.

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