Pancyclicity of Double Vertex Graph

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Abstract

The *double vertex graph* $U_2(G)$ of a graph G of order $n \geq 2$ is the graph whose vertex set consists of all $(n\ 2\ )$ unordered pairs of vertices of G and where two vertices $\{a, b\}$ and $\{c, d\}$ are adjacent if and only if $|\{a, b\} \cap \{c, d\}| = 1$ and if $a = c$, then b and d are adjacent in G. In this paper, we discuss some properties of $U_2(G)$ related to degree and distances. We also have obtained graphs G whose double vertex graphs contain G as a subgraph.

Keywords: Double vertex graphs, Degree, Distance,

2010 AMS Classification: *05C07, 05C12, 05C38.*

1. Introduction

All graphs G considered here are simple, finite and connected of order $n \geq 2$ and size m. Let $V(G)$ and $E(G)$ represent the vertex set and edge set of G, respectively. Chartrand introduced the term graph-valued function for any kind of rule or procedure which yeilds a unique graph from a given graph. In literature, there are many graph-valued functions studied,the earliest being line graph of a graph. The *double vertex graph* $U_2(G)$, is one such graph-valued function, introduced by Alavi.et.al[2].

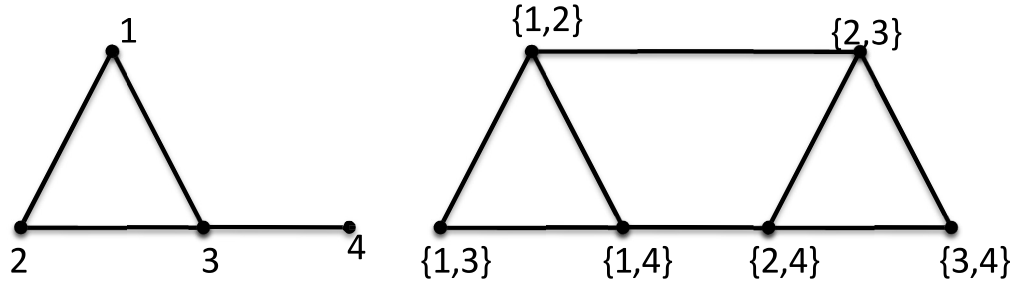

Figure 1: The graph G and its double vertex graph

For a graph G of order $n \geq 2$, $U_2(G)$ is the graph whose vertex set consists of all $(n\ 2\ )$ unordered pairs from $V(G)$ and where two vertices $\{a, b\}$ and $\{c, d\}$ are adjacent if and only if $|\{a, b\} \cap \{c, d\}| = 1$ and if $a = c$, then b and d are adjacent in G. The properties such as connectivity, planarity and traversability of double vertex graphs are well studied in [2,3,4].

In this paper, we characterize double vertex graphs that are pancyclic and also obtain pancyclicity of double vertex graphs of cycle

For undefined terminologies and notations refer to the text by G.Chartrand[1].

2. Preliminaries

A vertex $\{x, y\}$ of $U_2(G)$ is called a line pair or a non line pair according as x and y are adjacent or non adjacent in G. Then the vertex set of $U_2(G)$ can be partitioned into sets U and W where U is the set of line pairs and W the set of nonline pairs.

Theorem 2.1. $U_2(G)$ is connected if and only if G is connected.

Theorem 2.2. For any vertex $\{x, y\}$ of $U_2(G)$,

$$\deg\{x,y\} = \begin{cases} \deg x + \deg y - 2 & if\ \ xy \in E(G) \\ \deg x + \deg y & otherwise \end{cases}$$

Theorem 2.3([2]). If G is a hamiltonian graph of order $n \geq 4$ then $U_2(G)$ is hamiltonian if and only if a Hamiltonian cycle of G has an odd chord.

Remark 2.4 It is clear from above description of double vertex graph of cycles that any four vertices of the form $\{v_h, v_k\}, \{v_h, v_{k+1}\}, \{v_{h+1}, v_{k+1}\}, \{v_{h+1}, v_k\}$ where $1 \leq h \leq n-1$, $3 \leq k \leq n-2$ of $U_2(C_n)$ induces a four cycle in $U_2(C_n)$. So the shortest cycle of $U_2(C_n)(n \geq 4)$ is of order four.

3. Results

We begin with the following result:

A graph of order n is pancyclic if it has cycles of length 3 to n. By theorem 2.3 for a hamitonian graph G of order $n \geq 4$, $U_2(G)$ is hamiltonian if and only if hamiltonian cycle of G has an odd chord or $n = 5$(A chord $(1, k)$ is called odd chord when k is odd). We extend this result to prove pancyclicity of $U_2(G)$. Let $1,2,3, \ldots, n$ be the vertices of the hamiltonian cycle of G. For $U_2(G)$ to be pancyclic the smallest cycle of $U_2(G)$ must be a triangle. Since $U_2(G)$ has a triangle if and only if G has a triangle, we need to add a chord to the hamiltonian cycle of G which induces triangle in G. The possible chords which induces triangle in G are $(1,3)$ and $(1, n-1)$. Without loss of generality let us add the chord $(1, n-1)$ to the hamitonian cycle of G to prove the pancyclicity of $U_2(G)$.

Theorem 3.1: For a Hamiltonian graph G of order $n \geq 4$, $U_2(G)$ is pancyclic if hamiltonian cycle of G has a chord $(1, n-1)$.

Proof: Let $H: 1,2,3, \ldots, n$ be the hamiltonian cycle of G. Add the chord $(1, n-1)$ to H such that H' is hamiltonian cycle of G with a chord. Then $W = U_2(H)$ and $W' = U_2(H')$ are induced subgraphs of $U_2(G)$. Adding a chord to H adds $n-2$ edges to W and we call each such edge as "bridge edge". We prove $U_2(H')$ is hamiltonian and hence pancyclic for following cases of n.

Case 1 n is odd.

For any vertex $\{i, j\}$ in $W = U_2(H)$ the distance $d_G(i, j)$ between i and j in G is in the range of $1 \leq d_G(i, j) \leq \frac{n-1}{2}$. Partition the vertices of W into disjoint cycles based on the distance as

n is odd	Cycles of $U_2(H)$	Distance

$\frac{n-1}{2}$ **is even**	Q_k: $\{1,2k\},\{1,2k+1\},\{2,2k+1\},\ldots,\{2k-1,n-1\},\{2k-1,n\},\{2k,n\}$; $1 \le k \le \frac{n-1}{4}$	$d_G(u,v) \in \{2k-1,2k\}$
$\frac{n-1}{2}$ **is odd**	Q_k: $\{1,2k\},\{1,2k+1\},\{2,2k+1\},\ldots.,\{2k-1,n-1\},\{2k-1,n\},\{2k,n\}$; $1 \le k \le \frac{n-3}{4}$ $Q_{\frac{n+1}{4}}$: $\{1,\frac{n+1}{2}\},\{1,\frac{n+3}{2}\},\{2,\frac{n+3}{2}\},\ldots.,\{\frac{n-1}{2},n-1\},\{\frac{n-1}{2},n\},\{\frac{n+1}{2},n\}$	$d_G(u,v) \in \{2k-1,2k\}$ $d_G(u,v) = 2k-1$

Table 1 Partition of $U_2(H)$ for odd n

Depicted in table 1. We use bridge edges to go between cycles to form a Hamiltonian cycle of W'.

Subcase 1 $\frac{n-1}{2}$ is even.

Each cycle $Q_k : 1 \le k \le \frac{n-1}{4}$ induced by the vertices $\{u,v\}$ of W where $d_G(u,v) \in \{2k-1,2k\}$ is of order $2n$. In each cycle Q_k there are two consecutive vertices $\{1,2k\}$ and $\{1,2k+1\}$ which join to two consecutive vertices $\{2k,n-1\}$ and $\{2k+1,n-1\}$ of Q_{k+1} respectively by means of two bridge edges to form hamiltonian cycle of W'.

Subcase 2 $\frac{n-1}{2}$ is odd.

In this case each cycle $Q_k : 1 \le k \le \frac{n-3}{4}$ induced by the vertices $\{u,v\}$ where $d_G(u,v) \in \{2k-1,2k\}$ is of order $2n$ and $Q_{\frac{n+1}{4}}$ is of order n is induced by the vertices $\{u,v\}$ of W where $d_G(u,v) = 2k-1$. In each Q_k there are two consecutive vertices $\{1,2k\}$ and $\{1,2k+1\}$ which join two consecutive vertices of $\{2k,n-1\}$ and $\{2k+1,n-1\}$ of Q_{k+1} by means of bridge edges to form a Hamiltonian cycle of W'.

Case 2 n is even.

For any vertex $\{i,j\}$ in W the distance $d_G(i,j)$ is in the range of $1 \le d_G(i,j) \le \frac{n}{2}$ Unlike case 1, here we partition the vertices of W' into disjoint cycles based on the nature of $\frac{n}{2}$ as depicted in table 2.

Subcase 1 $\frac{n}{2}$ is odd

Each cycle $Q_k : 1 \le k \le \frac{n-2}{4}$ of order $2(n-1)$ is induced by the vertices $\{u,v\}$ where $d_G(u,v) \in \{2k-1,2k\}$. Further the remaining vertices $\{\{j,n\}: 1 \le j \le n-1\}$ of W' induces a cycle Q of order $n-1$ in W' where $d_G(j,n) = j$ for $1 \le j \le \frac{n}{2}$ and $n-j$ for $\frac{n}{2}+1 \le j \le n$. In each Q_k there are two consecutive vertices $\{1,2k\}$ and $\{1,2k+1\}$ joining to two consecutive vertices $\{2k,n\}$ and $\{2k+1,n\}$ of cycle Q to form the Hamiltonian cycle of W'.

Subcase 2 $\frac{n}{2}$ is even.

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In this case, each cycle $Q_k: 1 \leq k \leq \frac{n-4}{4}$ of order $2(n-1)$ is induced by the vertices $\{u, v\}$ of W' where $d_G(u, v) \in \{2k-1, 2k\}$ and the cycle $Q_{\frac{n}{4}}$ of order $n-1$ is induced by the vertices $\{u, v\}$ of W' where $d_G(u, v) = \frac{n}{2} - 1$. The remaining $n-1$ vertices of the form $\{\{j, n\}: 1 \leq j \leq n-1\}$ of W' induces a cycle Q where $d_G(j, n) = j$ for $1 \leq j \leq \frac{n}{2}$ and $n-j$ for $\frac{n}{2}+1 \leq j \leq n$. In each Q_k there are two consecutive vertices $\{1, 2k\}$ and $\{1, 2k+1\}$ which are adjacent to two consecutive vertices $\{2k, n\}$ and $\{2k+1, n\}$ of the cycle Q to form a Hamiltonian cycle of W'.

n **is even**	**Cycles of** $U_2(H')$	**Distance**
$\frac{n}{2}$ **is odd**	Q_k: $\{1,2k\}, \{1,2k+1\}, \{2,2k+1\}, \ldots, \{2k-1, n-1\}, \{2k-1, n-1\}, \{2k, n-1\}$; $1 \leq k \leq \frac{n-2}{4}$ Q: $\{1, n\}, \{2, n\}, \{3, n\}, \ldots, \{n-3, n\}, \{n-2, n\}, \{n-1, n\}$	$d_G(u, v) \in \{2k-1, 2k\}$ $d_G(j, n) = \{j \quad 1 \leq j \leq \frac{n}{2} \; n-j \quad \frac{n}{2}+1 \leq j \leq n$
$\frac{n}{2}$ **is even**	Q_k: $\{1,2k\}, \{1,2k+1\}, \{2,2k+1\}, \ldots, \{2k-1, n-2\}, \{2k-1, n-1\}, \{2k, n-1\}$; $1 \leq k \leq \frac{n-4}{4}$ $Q_{\frac{n}{4}}$: $\{1, \frac{n}{2}\}, \{1, \frac{n+2}{2}\}, \{2, \frac{n+2}{2}\}, \ldots, \{\frac{n-2}{2}, n-2\}, \{\frac{n-2}{2}, n-1\}, \{\frac{n}{2}, n-1\}$ Q: $\{1, n\}, \{2, n\}, \{3, n\}, \ldots, \{\frac{n}{2}, n\}, \{\frac{n+2}{2}, n\}, \{n-3, n\}, \{n-2, n\}, \{n-1, n\}$	$d_G(u, v) \in \{2k-1, 2k\}$ $d_G(u, v) = \frac{n}{2} - 1$ $d_G(j, n) = \{j \quad 1 \leq j \leq \frac{n}{2} \; n-j \quad \frac{n}{2}+1 \leq j \leq n$

Table 2 Partition of $U_2(H')$ for even n

In all the cases discussed above, W' has a hamiltonian cycle which implies a hamiltonian cycle in $U_2(G)$.

Further, to prove pancyclicity of $U_2(G)$, successively remove the vertices from hamiltonian cycle of $U_2(G)$ in order to get cycles of smaller lengths. The vertex to be removed must be so choosen that the end vertices of the accompanying edges of the removed vertex must be adjacent. Then we have following cases:

Case 1 n is odd

Subcase 1 $\frac{n-1}{2}$ is even.

Remove the vertices $\{1, 2k\}, \{2k-1, n-1\}$ from each the cycle $Q_k: 1 \leq k \leq \frac{n-5}{4}$ and $\{\frac{n-3}{2}, n-1\}$ from the cycle $Q_{\frac{n-1}{4}}$ succesively to get cycles of order $(n\,2) - 1, (n\,2) - 2, \ldots, \frac{(n-1)^2}{2} + 1$. Further, the cycles of order $4, 6, 8, 10, \ldots, \frac{(n-1)^2}{2}$ are obtained by successively inserting the two adjacent vertex pairs between two adjacent vertices of a four cycle as illustrated in Table 2.

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The longest cycle thus obtained is of order $2n-4$ given by $C_{2n-4} \cong \{1,3\}-\{1,4\}-\{1,5\}-\{1,6\}-\{1,7\}-\cdots-\{1,n\}-\{2,n\}-\cdots-\{2,7\}-\{2,6\}-\{2,5\}-\{2,4\}-\{2,3\}$. To this cycle insert vertex pairs of the form $\{3,k\}-\{3,k-1\}$ between the vertex pair $\{2,k\}-\{2,k-1\}$ of C_{2n-4} where $k=5,7,9,\dots,n$ to get cycle of order $2n-2$. Similarly insert vertex pair $\{4,n\}-\{4,n-1\}$ between the vertices $\{3,n\},\{3,n-1\}$ of the cycle C_{2n-2} to get a cycle C_{2n}. Continue in the same manner till we have a cycle of order $\frac{(n-1)^2}{2}$ and the vertices $\{1,2\},\{4,5\},\{6,7\},\{8,9\},\dots,\{n-1,n\}$ do not lie on this cycle.

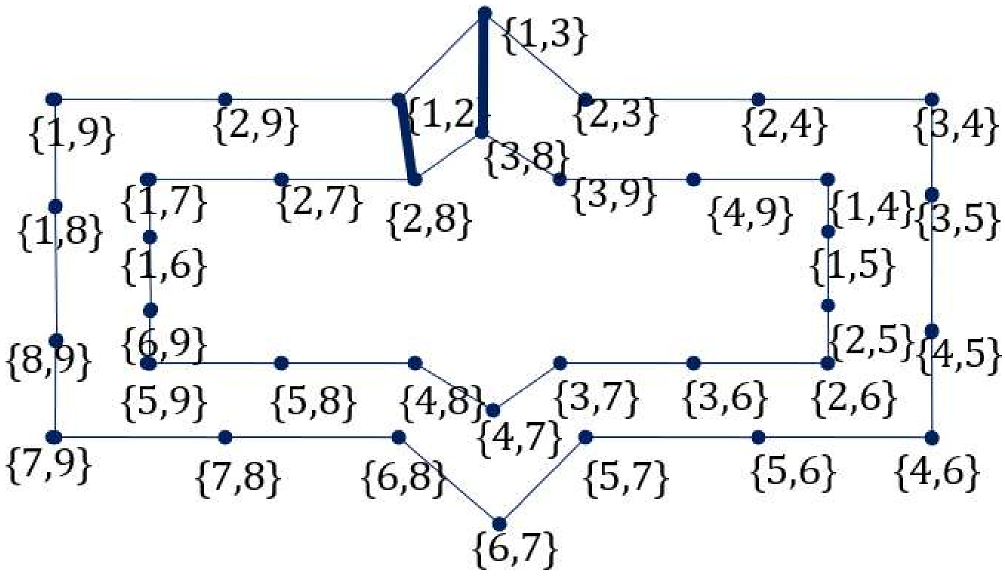

Figure 2: Hamiltonian cycle of $U_2(C_9)$ with chord (1,8)

Now to obtain cycles of order $3,5,7,9,\dots,2n-3$ in $U_2(H)$ we follow the same method as above.We start from the cycle $\{1,n-1\}-\{1,n\}-\{2,n\}-\{3,n\}-\cdots-\{n-1,n\}-\{1,n-1\}$

of order n and insert two adjacent vertex pairs between two adjacent vertices of this cycle as depicted in table 3.

Some even Cycles of $U_2(G)$	**Order**
$\{1,3\}-\{1,4\}-\{2,4\}-\{2,3\}$	4
$\{1,3\}-\{1,4\}-\{1,5\}-\{2,5\}-\{2,4\}-\{2,3\}$	6
$\{1,3\}-\{1,4\}-\{1,5\}-\{1,6\}-\{2,6\}-\{2,5\}-\{2,4\}-\{2,3\}$	8
$\{1,3\}-\{1,4\}-\{1,5\}-\{1,6\}-\{1,7\}-\{2,7\}-\{2,6\}-\{2,5\}-\{2,4\}-\{2,3\}$	10
....	
$\{1,3\}-\{1,4\}-\{1,5\}-\{1,6\}-\{1,7\}-\cdots-\{1,n\}-\{2,n\}-\cdots-\{2,7\}-\{2,6\}-\{2,5\}-\{2,4\}-\{2,3\}$	$2n-4$

Table 3: cycles of even order in $U_2(G)$

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The longest cycle thus obtained will be C_{3n-8}. To this cycle insert $\{4, n-1\}, \{5, n-1\}$ between $\{4, n\}$ and $\{5, n\}$ to get cycle of length $3n-6$. Similarly insert $\{4, n-2\}, \{5, n-2\}$ between $\{4, n-1\}$ and $\{5, n-1\}$ to get cycle of length $3n-4$. Continue till we insert $\{4,6\}, \{5,6\}$ between $\{4,7\}, \{5,7\}$ of the cycle obtained in previous step. So we have inserted $2n-12$ vertices of the form $\{4, k\}, \{5, k\}$ between $\{4, k+1\}$ and $\{5, k+1\}$ of the cycle obtained in previous step where $k = n-1, n-2, n-3, \ldots, 6$. We have a cycle of order $5n-20$.

Further we insert $2n-16$ vertices of the form $\{6, z\}, \{7, z\}$ between $\{6, z+1\}$ and $\{7, z+1\}$ of the cycle obtained in previous step where $z = n-1, n-2, n-3, \ldots, 8$ to get cycle of length $7n-36$. Continue till we insert $\{n-3, n-1\}, \{n-2, n-1\}$ between $\{n-3, n\}$ and $\{n-2, n\}$ of the cycle obtained in previous step to get cycle of length $n + 2n - 8 + 2n - 12 + 2n - 16 + \cdots + 2n - (2n-2) = n + \frac{(n-3)^2}{2}$. Further insert $\{1, n-2\}, \{1, n-3\}$ between the vertices $\{2, n-2\}, \{2, n-3\}$ of the cycle obtained in previous step to get a cycle of order $n + \frac{(n-3)^2}{2} + 2$. Then

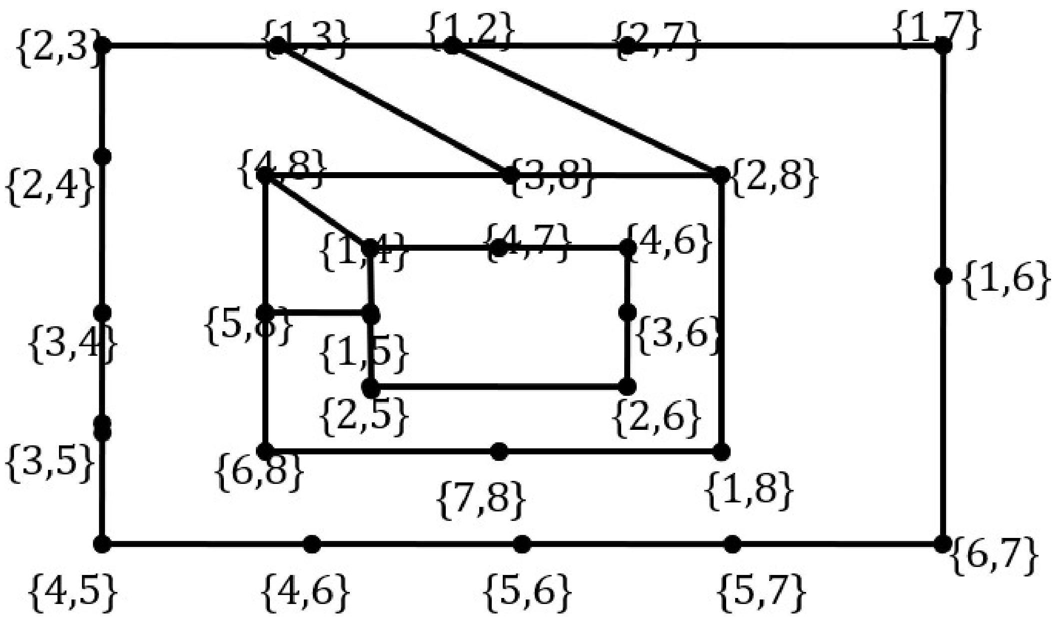

Figure 3: Hamiltonian cycle of $U_2(C_8)$ with chord (1,7)

to this cycle we insert $\{1, n-4\}, \{1, n-5\}$ between the vertices $\{2, n-4\}, \{2, n-5\}$ to get a cycle of order $n + \frac{(n-3)^2}{2} + 2 + 2$.

Continue till we insert $n-5$ vertices of the type $\{1, z\}, \{1, z-1\}$ between the vertices $\{2, z\}, \{2, z-1\}$ of the cycle obtained in previous step where $z = n-2, n-4, \ldots, 5$ we get a cycle of order$n + \frac{(n-3)^2}{2} + n - 5 = \frac{(n-1)^2}{2} - 1$. Thus we have proved that $U_2(G)$ has cycles of odd order $n, n+2, n+4, n+5, \ldots, \frac{(n-1)^2}{2} - 1$. We are left with only cycles of order $3,5,7, \ldots, n-2$ to prove $U_2(G)$ pancyclic. We start from a triangle$\{1, n-1\} - \{1, n\} - \{n-1, n\}$. Further insert $2(n-3)$ vertices of the form $\{\{k, n\}: k = n-2, n-$

$3, \dots, 2\}$ and $\{1, k\}$ in pairs between $\{n-1, n\} - \{1, n-1\}$ of the triangle simultaneously to get cycles of order $5,7,9, \dots, 2n-3$ respectively.

Subcase 2 $\frac{n-1}{2}$ is odd

Proof: It is same as subcase 1.

Case 2 n is even

Subcase $\frac{n}{2}$ is odd

Now we remove vertices of the type $\{1,2k\}$ from cycle Q_k till we get a cycle of order $p = (n\ 2\ ) - \frac{n-2}{4}$. We continue further based on the nature of $\frac{n-2}{4}$.

Some Odd Cycles of $U_2(G)$	**Order**
$\{1, n-1\} - \{1, n\} - \{2, n\} - \{3, n\} - \cdots - \{n-1, n\} - \{1, n-1\}$	n
$\{1, n-1\} - \{1, n\} - \{2, n\} - \{2, n-1\} - \{3, n-1\} - \{3, n\} - \cdots - \{n-1, n\} - \{1, n-1\}$	$n+2$
$\{1, n-1\} - \{1, n\} - \{2, n\} - \{2, n-1\} - \{2, n-2\} - \{3, n-2\} \{3, n-1\} - \{3, n\} - \cdots - \{n-1, n\} - \{1, n-1$	$n+4$
$\{1, n-1\} - \{1, n\} - \{2, n\} - \{2, n-1\} - \{2, n-2\} - \{2, n-3\} - \{3, n-3\} - \{3, n - 2\} \{3, n-1\} - \{3, n\} - \cdots - \{n-1, n\} - \{1, n-1$	$n+6$
....	
$\{1, n-1\} - \{1, n\} - \{2, n\} - \{2, n-1\} - \{2, n-2\} - \{2, n-3\} - \cdots \{2,4\} - \{3,4\} - \cdots - \{3, n-3\} - \{3, n-2\} \{3, n-1\} - \{3, n\} - \cdots - \{n-1, n\} - \{1, n-1$	$3n-8$

Table 4: Some odd cycles of $U_2(G)$

Subcase 1: $\frac{n-2}{4}$ is even.

Insert $\{1,2\,(\frac{n-2}{4})\}$ and remove $\{2, n-1\}$,$\{2, n\}$ simultaneously to get cycle of length $p-1$. Then add $\{1,2(\frac{n-2}{4} - 1)\}$ back into the cycle and remove the vertex $\{4, n-1\}$ and $\{4, n\}$ simultaneously to get cycle of length $p-2$. Continue till we add $\{1,2(\frac{n-2}{4} - 1)\}$ back into the cycle $Q_{\frac{n+6}{8}}$ and remove $\{2\,(\frac{n-2}{8}), n-1\}$ and $\{2\,(\frac{n-2}{8}), n\}$ from $Q_{\frac{n-2}{8}}$ and Q respectively to get a cycle of length $p - (\frac{n-2}{8})$. Finally we remove the vertices of the type $\{1,2k\}$ from each of the cycles $Q_k: k = \frac{n-2}{4}, \frac{n-2}{4} - 1, \frac{n-2}{4} - 2, \dots, \frac{n+6}{8}$ till we have a cycle of length $\frac{n(n-2)}{2} + 1$.

Subcase 2. $\frac{n-2}{4}$ is odd.

Insert $\{1,2\,(\frac{n-2}{4})\}$ to the cycle C_p and remove $\{2, n-1\}$ and $\{2, n\}$ simultaneously to get cycle of length $p-1$. Then insert $\{1,2\,(\frac{n-2}{4}-1)\}$ and remove $\{4, n-1\}$ and $\{4, n\}$ simultaneously to get cycle of length $p-2$. Continue till we insert $\{1,2\,(\frac{n+10}{8})\}$ back to $C_{\frac{n+10}{8}}$ and then remove $\{2\,(\frac{n-6}{8}), n-1\}$ from $C_{\frac{n-6}{8}}$ and $\{2\,(\frac{n-6}{8}), n\}$ from Q. Further remove the vertices $\{1,2k\}$ from each of the cycle C_k where $k: \frac{n-2}{4}, \frac{n-2}{4}-1, \dots, \frac{n+10}{8}$ one by one to get cycle of length $\frac{n(n-2)}{2}+2$. The cycle of length $\frac{n(n-2)}{2}+1$ can be obtained by adding the vertex $\{1, n-1\}, \{1, n\}$ to the cycle of length $\frac{n(n-2)}{2}$ obtained in theorem 2.4

Subcase 2 $\frac{n}{2}$ is even.

We remove vertices of the type $\{1,2k\}$ each from cycle Q_k till we have a cycle of order $l = \frac{n}{4}(2n-3)$. Further removal of vertices to get cycles of next order depends on the nature of $\frac{n-4}{4}$.

Subcase 1 $\frac{n-4}{4}$ is even

Insert $\{1,2\frac{n}{4}\}$ to C_l and simultaneously remove $\{2, n-1\}, \{2, n\}$ to get cycle of length $l-1$. Then insert $\{1,2(\frac{n}{4}-1)\}$ and simultaneously remove $\{4, n-1\}$, $\{4, n\}$ to get cycle of length $l-2$. Continue the process till we insert a vertex to the cycle of length $l-(\frac{n-4}{8})-1$ and remove the vertices $\{2\,(\frac{n-4}{8}), n-1\}$ and $\{2\,(\frac{n-4}{8}), n\}$ to get cycle of length $l-(\frac{n-4}{8})$. Now from the cycle of length $l-(\frac{n-4}{8})$ we remove vertices of the type $\{1,2\frac{n}{4}\}, \{1,2\,(\frac{n}{4}-1)\}, \{1,2\,(\frac{n}{4}-2)\}, \dots, \{1,2(\frac{n}{4}-(\frac{n-4}{8}))\}$ one by one till we get cycle of length $l-(\frac{n-4}{8})-(\frac{n-4}{8}) = \frac{n}{4}(2n-3) - 2\,(\frac{n-4}{8}) = \frac{n(n-2)}{2}+1$

Subcase 2 $\frac{n-4}{4}$ is odd.

Insert $\{1,2\frac{n}{4}\}$ and simultaneously remove $\{2, n-1\}$ and $\{2, n\}$ to get cycle of length $l-1$. Then insert the vertex $\{1,2(\frac{n}{4}-1)\}$ and simultaneously remove $\{4, n-1\}$ and $\{4, n\}$ to get cycle of length $l-2$. Continue the process till we insert $\{1,2(\frac{n}{4}-(\frac{n-8}{8}))\}$ to the cycle of length $l-(\frac{n}{8})-1$ and remove the vertex $\{2\,(\frac{n-4}{8}), n-1\}$ and $\{2\,(\frac{n-4}{8}), n\}$ to get cycle of length $l-(\frac{n-4}{8})$. Now from cycle of length $l-(\frac{n}{8})$ we remove $\frac{n-4}{4}$ vertices of the type $\{1,2(\frac{n}{4}-(\frac{n-4}{3}))\}, \{1,2\,(\frac{n}{4}-(\frac{n-4}{3}))+1\}, \{1,2\,(\frac{n}{4}-(\frac{n-4}{3}))+3\}, \dots, \{1,2(\frac{n}{4}-2)\}$ one by one till we get cycle of length $l-(\frac{n-4}{4}) = \frac{n}{4}(2n-3)-(\frac{n-4}{4}) = \frac{n(n-2)}{2}+1$.

From all the above cases it is clear that $U_2(G)$ has cycles of order $(n\ 2\ ), (n\ 2\ ) - 1, (n\ 2\ ) - 2, \dots, \frac{n(n-2)}{2} + 1$. From theorem 1.4.1 and by case 1, we can obtain cycles of order $4,6,8, \dots, \frac{n(n-2)}{2}$ in $U_2(G)$. Therefore we need cycles of order $3,5,7,9, \dots, n-1, n+1, n+3, \dots, \frac{n(n-2)}{2} - 1$ to prove $U_2(G)$ pancyclic. We start from a triangle $\{1, n-1\} - \{n-1, n\} - \{1, n\} - \{1, n-1\}$ in W'. To this triangle between $\{j-1, n\}, \{j-1, n-1\}$ we insert vertex pairs of the form $\{j, n\} - \{j, n-1\}$ where $2 \le j \le n-2$ to get cycles of order $5,7,9, \dots, 2n-3$. To this cycle insert $\{n-4, n-2\}, \{n-3, n-2\}$ between $\{n-4, n-1\}$ and $\{n-3, n-1\}$ to get cycle of order $2n-1$. Then we insert vertices of the form $\{j, k\}, \{j+1, k\}$ in pairs between $\{j, n-1\}, \{j+1, n-1\}$ where $j = n-4, n-2, \dots, 2$ and $n-2 \le k \le j+2$ to get a cycle of order $2n - 3 + \frac{(n-4)^2}{2}$ Further insert $n-6$ vertices $\{1, j\}, \{1, j+1\}$ in pairs between the vertices $\{2, j\}, \{2, j+1\}$ of the cycle obtained previously where $j = 4,6, \dots, n-4$ to get a cycle of length $2n - 3 += \frac{n(n-2)}{2} - 1$.

Example 1: Pancyclicity of $U_2(C_7)$ with chord (1,6)

$V(U_2(C_7)) = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}\}, |V(U_2(C_7))| = 21$

We construct cycles of even order starting from C_4 as discussed in theorem 3.1

- C_4: $\{1,3\} - \{1,4\} - \{2,4\} - \{2,3\}$
- C_6: $\{1,3\} - \{1,4\} - \{1,5\} - \{2,5\} - \{2,4\} - \{2,3\}$
- C_8: $\{1,3\} - \{1,4\} - \{1,5\} - \{1,6\} - \{2,6\} - \{2,5\} - \{2,4\} - \{2,3\}$
- C_{10}: $\{1,3\} - \{1,4\} - \{1,5\} - \{1,6\} - \{1,7\} - \{2,7\} - \{2,6\} - \{2,5\} - \{2,4\} - \{2,3\}$
- C_{12}: $\{1,3\} - \{1,4\} - \{1,5\} - \{1,6\} - \{1,7\} - \{2,7\} - \{2,6\} - \{2,5\} - \{3,5\} - \{3,4\} - \{2,4\} - \{2,3\}$
- C_{14}: $\{1,3\} - \{1,4\} - \{1,5\} - \{1,6\} - \{1,7\} - \{2,7\} - \{3,7\} - \{3,6\} - \{2,6\} - \{2,5\} - \{3,5\} - \{3,4\} - \{2,4\} - \{2,3\}$
- C_{16}: $\{1,3\} - \{1,4\} - \{1,5\} - \{1,6\} - \{1,7\} - \{2,7\} - \{3,7\} - \{4,7\} - \{4,6\} - \{3,6\} - \{2,6\} - \{2,5\} - \{3,5\} - \{3,4\} - \{2,4\} - \{2,3\}$
- C_{18}: $\{1,3\} - \{1,4\} - \{1,5\} - \{1,6\} - \{1,7\} - \{2,7\} - \{3,7\} - \{4,7\} - \{5,7\} - \{5,6\} - \{4,6\} - \{3,6\} - \{2,6\} - \{2,5\} - \{3,5\} - \{3,4\} - \{2,4\} - \{2,3\}$

Then hamiltonian cycle of $U_2(C_7)$ is

$$C_{21} \cong \{1,2\} - \{2,7\} - \{1,7\} - \{1,6\} - \{6,7\} - \{5,7\} - \{5,6\} - \{4,6\} - \{4,5\} - \{3,5\} - \{3,4\} - \{2,4\} - \{2,3\} - \{1,3\} - \{3,6\} - \{3,7\} - \{4,7\} - \{1,4\} - \{1,5\} - \{2,5\} - \{2,6\}$$

Removing the vertices from hamiltonian cycle as discused in theorem 3.1 we obtain

- $C_{20} \cong \{2,7\} - \{1,7\} - \{1,6\} - \{6,7\} - \{5,7\} - \{5,6\} - \{4,6\} - \{4,5\} - \{3,5\} - \{3,4\} - \{2,4\} - \{2,3\} - \{1,3\} - \{3,6\} - \{3,7\} - \{4,7\} - \{1,4\} - \{1,5\} - \{2,5\} - \{2,6\}$
- $C_{19} \cong \{2,7\} - \{1,7\} - \{6,7\} - \{5,7\} - \{5,6\} - \{4,6\} - \{4,5\} - \{3,5\} - \{3,4\} - \{2,4\} - \{2,3\} - \{1,3\} - \{3,6\} - \{3,7\} - \{4,7\} - \{1,4\} - \{1,5\} - \{2,5\} - \{2,6\}$

Further cycles of order 3,5,7,9,11,13,15,17 can be obtained as below:

- C_3: $\{1,6\} - \{1,7\} - \{6,7\}$
- C_5: $\{1,6\} - \{1,7\} - \{6,7\} - \{5,7\} - \{1,5\}$
- C_7: $\{1,6\} - \{1,7\} - \{6,7\} - \{5,7\} - \{4,7\} - \{1,4\} - \{1,5\}$
- C_9: $\{1,6\} - \{1,7\} - \{6,7\} - \{5,7\} - \{4,7\} - \{3,7\} - \{1,3\} - \{1,4\} - \{1,5\}$
- C_{11}: $\{1,6\} - \{1,7\} - \{6,7\} - \{5,7\} - \{4,7\} - \{3,7\} - \{2,7\} - \{1,2\} - \{1,3\} - \{1,4\} - \{1,5\}$
- C_{13}: $\{1,6\} - \{1,7\} - \{6,7\} - \{5,7\} - \{4,7\} - \{3,7\} - \{2,7\} - \{1,2\} - \{1,3\} - \{1,4\} - \{1,5\}$
- C_{15}: $\{1,6\} - \{1,7\} - \{6,7\} - \{5,7\} - \{4,7\} - \{3,7\} - \{3,6\} - \{2,6\} - \{2,7\} - \{1,2\} - \{1,3\} - \{1,4\} - \{1,5\}$
- C_{17}: $\{1,6\} - \{1,7\} - \{6,7\} - \{5,7\} - \{4,7\} - \{3,7\} - \{3,6\} - \{3,5\} - \{2,5\} - \{2,6\} - \{2,7\} - \{1,2\} - \{1,3\} - \{1,4\} - \{1,5\}$

Thus $U_2(C_7)$ has induced cycles from length 3 to 21 hence $U_2(C_7)$ is pancyclic.

4. Conclusion

In this paper, we have chararcterised double vertex graph of a graph G that are pancyclic..Also,we have showed that double vertex graph of cycle graph with a chord is pancyclic..

5. References

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